SpTp: MATH 1199 - EXAM 3 FALL 2019

SOLUTION

Friday, 22 November
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

Formulas

ML-inequality: Let C be a contour. If $f(z) \leq M$ on the contour C and L denotes the length of C, then

$$\left| \int_C f(z)\mathrm{d}z \right| \leq ML.$$

Cauchy Integral Theorem: If C is a simple closed contour and $f\colon U \subset \mathbb{C} \to \mathbb{C}$ is complex-differentiable everywhere in the interior of C, then $\displaystyle\int_C f(z)\mathrm{d}z = 0$.

Cauchy Integral Formula: If $f\colon U \to \mathbb{C}$ is complex-differentiable on the interior of a simpled closed contour C, then for any $\zeta \in U$,

$$f(\zeta) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-\zeta}\mathrm{d}z.$$

If a function f has a blow-up point at z_0 and a Laurent series of the form

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k,$$

then we say that the residue of f at $z = z_0$ is the coefficient c_{-1}, i.e. we write

$$\operatorname*{Res}_{z=z_0} f(z) = c_{-1}.$$

Cauchy Residue Theorem: If C is a contour and the poles $z_0, \ldots, z_n$ of a function f are in the interior of C, then

$$\int_C f(z)\mathrm{d}z = 2\pi i \sum_{k=0}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Inversion of Laplace transform: if F has finitely many poles and the contour C surrounds all of them, then we may compute the inverse Laplace transform of F as

$$\mathscr{L}^{-1}\{F\}(t) = \frac{1}{2\pi i} \int_C e^{zt} F(z)\mathrm{d}z$$

1. (14 points) Consider the contour C consisting of the upper half of the circle $|z| = 3$. Use the ML inequality to bound the following:
$$\left|\int_C \frac{z-2}{z^2-7}\mathrm{d}z\right|.$$
Solution: Bound the numerator with the triangle inequality:
$$|z-2| \leq |z| + |2| = 3 + 2 = 5,$$
and bound the denominator with the reverse triangle inequality:
$$|z^2-7| \geq ||z|^2 - |7|| = |3^2 - 7| = 2,$$
hence
$$\frac{1}{|z^2-7|} \leq \frac{1}{2}.$$
Therefore,
$$\left|\frac{z-2}{z^2-7}\right| \leq \frac{5}{2} =: M.$$
The length of the semicircle C is $L := 3\pi$. Therefore, by the ML-inequality
$$\left|\int_C \frac{z-2}{z^2-7}\mathrm{d}z\right| \leq ML = \frac{15\pi}{2}.$$

2. (15 points) The Bessel function of order five, $J_5 \colon \mathbb{C} \to \mathbb{C}$, is a complex-differentiable function given by the infinite series $J_5(z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+5}}{k!(5+k)!2^{2k+5}}$. Find the residue at $z = 0$ of the function $g(z) = z^4 J_5\left(\frac{1}{z}\right)$.
Solution: Consider the series for g:
$$\begin{aligned} g(z) &= z^4 J_5\left(\frac{1}{7}\right) \\ &= z^4 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(5+k)!2^{2k+5}z^{2k+5}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(5+k)!2^{2k+5}z^{2k+1}} \\ &= \frac{(-1)^0}{0!5!2^5 z} + \frac{(-1)^1}{1!6!2^7 z^3} + \dots \end{aligned}$$
Since the residue is the coefficient of $\frac{1}{z}$, we conclude that
$$\operatorname*{Res}_{z=0} g(z) = \frac{1}{5!2^5}.$$

3. (15 points) Use the inversion integral to find the inverse Laplace transform of $F(z) = \dfrac{1}{(z+2)^2}$
Solution: The poles of this function come from setting $(z+2)^2 = 0$, i.e. $z = -2$, a pole of order 2. Therefore, for a closed contour C with -2 interior to it,
$$\begin{aligned} \mathscr{L}^{-1}\{F\}(t) &= \frac{1}{2\pi i}\int_C e^{tz}F(z)\mathrm{d}z \\ &\overset{\text{res thm}}{=} \frac{1}{2\pi i}2\pi i \operatorname*{Res}_{z=-2} \frac{e^{tz}}{(z+2)^2} \\ &\overset{\phi(z)=e^{zt}}{=} \frac{\phi'(-2)}{1!} \\ &\overset{\phi'(z)=te^{zt}}{=} te^{-2t}. \end{aligned}$$

4. (24 points) In this problem, you will find series, centered at zero, for the function $f(z) = \frac{1}{z-1} + \frac{1}{z-5}$.

(a) (8 points) Find a series for f valid in the region $|z| < 1$.
Solution: For $|z| < 1$,

$$(i) \qquad \frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{k=0}^{\infty} z^k, \qquad |z| < 1,$$

and

$$(ii) \qquad \frac{1}{z-5} = -\frac{1}{5}\frac{1}{1-\frac{z}{5}} = -\frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{z}{5}\right)^k = -\sum_{k=0}^{\infty}\frac{z^k}{5^{k+1}}, \qquad \left|\frac{z}{5}\right| < 1 \leftrightarrow |z| < 5.$$

Therefore, for $|z| < 1$, the following series is valid:

$$f(z) = -\sum_{k=0}^{\infty} z^k - \sum_{k=0}^{\infty}\frac{z^k}{5^{k+1}}.$$

(b) (8 points) Find a series for f valid in the region $1 < |z| < 5$.
Solution: For $1 < |z| < 5$, the series (ii) is still valid, but the series (i) is no longer valid. Therefore,

$$(iii) \qquad \frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}}, \qquad \left|\frac{1}{z}\right| < 1 \leftrightarrow |z| > 1$$

Therefore, for $1 < |z| < 5$, the following series is valid:

$$f(z) = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}} - \sum_{k=0}^{\infty}\frac{z^k}{5^{k+1}}.$$

(c) (8 points) Find a series for f valid in the region $|z| > 5$.
Solution: For $|z| > 5$, the series (iii) is remains valid, but the series (ii) is no longer valid. Therefore,

$$(iv) \qquad \frac{1}{z-5} = \frac{1}{z}\frac{1}{1-\frac{5}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{5^k}{z^k} = \sum_{k=0}^{\infty}\frac{5^k}{z^{k+1}}, \qquad \left|\frac{5}{z}\right| < 1 \leftrightarrow |z| > 5.$$

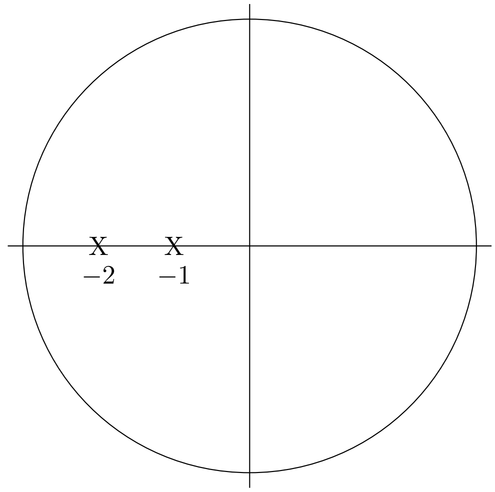
Therefore, for $|z| > 5$, the following series is valid:

$$f(z) = \sum_{k=0}^{\infty}\frac{1}{z^{k+1}} + \sum_{k=0}^{\infty}\frac{5^k}{z^{k+1}}.$$

5. (32 points) Contour integrals. For each part, ***draw a sketch of all relevant blow-up points and the contour*** and calculate the integral in any way you wish.

(a) (8 points) Let C be the contour $|z| = 3$ and calculate $\displaystyle\int_C \frac{1}{(z+2)(z+1)}\mathrm{d}z$.

Solution:

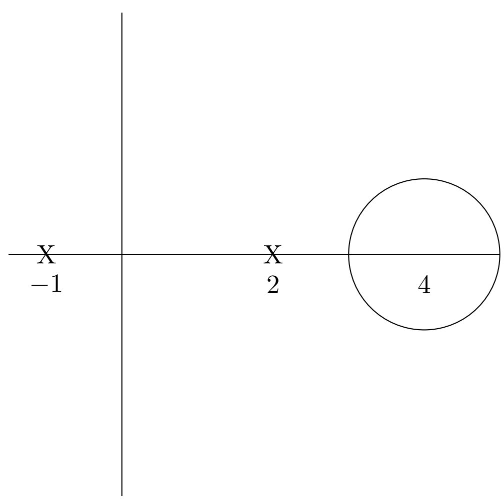


Both (order 1) poles appear interior to the contour, so we will use the residue theorem to compute the integral:

$$\begin{aligned}\int_C \frac{1}{(z+2)(z+1)}\mathrm{d}z &= 2\pi i \operatorname*{Res}_{z=-1} \frac{\frac{1}{z+2}}{z-(-1)} + 2\pi i \operatorname*{Res}_{z=-2} \frac{\frac{1}{z+1}}{z-(-2)} \\ &= 2\pi i \left(\frac{1}{-1+2}\right) + 2\pi i \left(\frac{1}{-2+1}\right) \\ &= 2\pi i\,(1 + -1) \\ &= 0.\end{aligned}$$

(b) (8 points) Let C be the contour $|z-4| = 1$ and calculate $\displaystyle\int_C \frac{1}{(z-2)(z+1)}\mathrm{d}z$.

Solution:

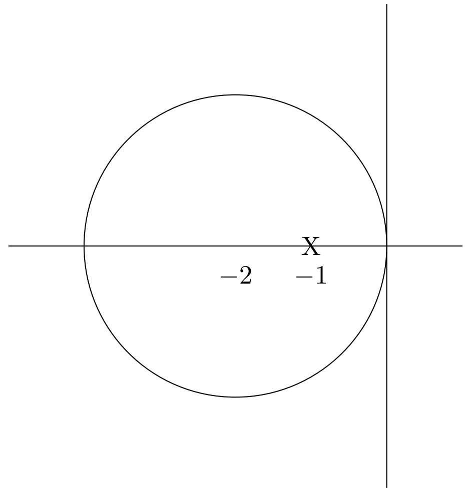


None of the poles appear interior to the contour. Therefore we may use the Cauchy integral theorem to say

$$\int_C \frac{1}{(z-2)(z+1)}\mathrm{d}z = 0.$$

(c) (8 points) Let C be the contour $|z+2| = 2$ and calculate $\displaystyle\int_C \frac{\sin(z)\cos(z)e^z}{z+1}\mathrm{d}z$.

Solution:

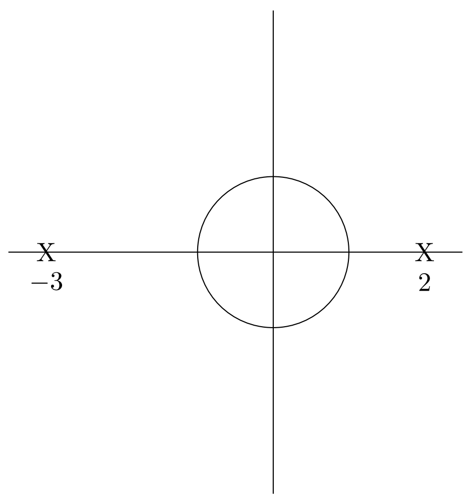


The simple pole -1 appears interior to the contour. Therefore we use the residue theorem to compute

$$\begin{aligned}\int_C \frac{\sin(z)\cos(z)e^z}{z+1}\mathrm{d}z &= 2\pi i \operatorname*{Res}_{z=-1} \frac{\sin(z)\cos(z)e^z}{z-(-1)} \\ &= 2\pi i \sin(-1)\cos(-1)e^{-1}.\end{aligned}$$

(d) (8 points) Let C be the contour $|z| = 1$ and calculate $\displaystyle\int_C \frac{e^z}{(z-2)(z+3)}\mathrm{d}z$.

Solution:



None of the poles appear interior to the contour. Therefore we may use the Cauchy integral theorem to compute

$$\int_C \frac{e^z}{(z-2)(z+3)}\mathrm{d}z = 0.$$