SpTp: MATH 1199 - EXAM 2 FALL 2019

SOLUTION

Friday, 18 October
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (18 points) Compute from the definition and write in the form $a + bi$.

   (a) (9 points) 3^{2+i}

   Solution: Compute

$$\begin{aligned} 3^{2+i} &= e^{(2+i)\log(3)} \\ &= e^{(2+i)[\ln(3)+2\pi i n]} \\ &= e^{[2\ln(3)-2\pi n]+[\ln(3)+4\pi n]i} \\ &= e^{2\ln(3)-2\pi n}\Big(\cos(\ln(3)+4\pi n)+i\sin(\ln(3)+4\pi n)\Big) \\ &= 9e^{-2\pi n}\Big(\cos(\ln(3)+4\pi n)+i\sin(\ln(3)+4\pi n)\Big) \end{aligned}$$

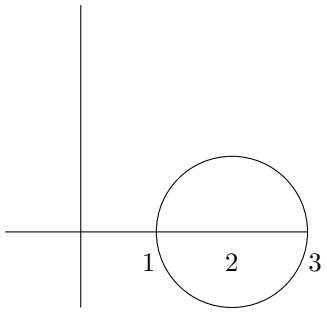
   (b) (9 points) $\text{Arccos}\left(\dfrac{\sqrt{3}}{2}\right)$

   Solution: Compute

$$\begin{aligned} \text{Arccos}\left(\frac{\sqrt{3}}{2}\right) &= \frac{1}{i}\text{Log}\left(\frac{\sqrt{3}}{2}+\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2-1}\right) \\ &= \frac{1}{i}\text{Log}\left(\frac{\sqrt{3}}{2}+\sqrt{-\frac{1}{4}}\right) \\ &= \frac{1}{i}\text{Log}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) \\ &= \frac{1}{i}\left[\ln\left(\left|\frac{\sqrt{3}}{2}+\frac{1}{2}i\right|\right)+i\text{Arg}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right] \\ &= \frac{1}{i}\left[\underbrace{\ln(1)}_{=0}+i\frac{\pi}{6}\right] \\ &= \frac{\pi}{6} \end{aligned}$$

2. (20 points) Draw the circle $|z-2|=1$ and explain what it maps to under the Möbius transformation $f(z)=\dfrac{1}{z}$.

   Solution:



   Let C denote this circle. By definition, $w \in f[C]$ if and only if $\dfrac{1}{w} \in C$, meaning

$$\left|\frac{1}{w}-2\right|=1.$$

   Multiplying this by $|w|$ yields

$$|1-2w|=|w|.$$

Square both sides to get

$$|1-2w|^2 = |w|^2.$$

Now using the relation $z\overline{z} = |z|^2$, we rewrite these moduli as

$$(1-2w)(1-2\overline{w}) = w\overline{w}.$$

Expand the left-hand side to get

$$1 - 2\overline{w} - 2w + 4w\overline{w} = w\overline{w},$$

which is algebraically equivalent to

$$w\overline{w} - \frac{2}{3}\overline{w} - \frac{2}{3}w = -\frac{1}{3}.$$

Adding $\left(-\frac{2}{3}\right)^2$ to both sides yields (note: we can drop the minus here because of the squaring)

$$w\overline{w} - \frac{2}{3}\overline{w} - \frac{2}{3}w + \left(\frac{2}{3}\right)^2 = -\frac{3}{9} + \frac{4}{9}.$$

Factoring the left-hand side yields

$$\overline{w}\left(w - \frac{2}{3}\right) - \frac{2}{3}\left(w - \frac{2}{3}\right) = \frac{1}{9},$$

yielding

$$\left(w - \frac{2}{3}\right)\left(\overline{w} - \frac{2}{3}\right) = \frac{1}{9}.$$

Therefore, applying the identity $z\overline{z} = |z|^2$ again yields

$$\left|w - \frac{2}{3}\right| = \frac{1}{9}.$$

This shows that the circle $|z-2| = 1$ maps to the circle $\left|z - \frac{2}{3}\right| = \frac{1}{9}$ under the given map.

3. (25 points) Do the following problems.

   (a) (8 points) Find the Möbius transformation $f(z)$ that maps $3 \mapsto 0$, $-7 \mapsto 1$, $\infty \mapsto \infty$.
   Solution:

$$f(z) = \left(\frac{z-3}{\cancel{\text{“}z-\infty\text{”}}}\right)\left(\frac{\cancel{\text{“}-7-\infty\text{”}}}{-7-3}\right) = -\frac{1}{10}(z-3)$$

   (b) (8 points) Find the Möbius transformation $g(z)$ that maps $1 \mapsto 0$, $-2 \mapsto 1$, and $3 \mapsto \infty$.
   Solution:

$$g(z) = \left(\frac{z-1}{z-3}\right)\left(\frac{-2-3}{-2-1}\right) = \left(\frac{z-1}{z-3}\right)\left(\frac{-5}{-3}\right) = \frac{5}{3}\left(\frac{z-1}{z-3}\right).$$

   (c) (9 points) Find the Möbius transformation that maps $1 \mapsto 3$, $-2 \mapsto -7$, and $3 \mapsto \infty$.
   Solution: Looking at what maps where shows that the map $h(z) = f^{-1}(g(z))$ will do the job, because

$$h(1) = f^{-1}(g(1)) = f^{-1}(0) = 3,$$

$$h(-2) = f^{-1}(g(-2)) = f^{-1}(1) = -7,$$

   and

$$h(3) = f^{-1}(g(3)) = f^{-1}(\infty) = \infty.$$

So, to find $f^{-1}(z)$, let $z = f(w)$ and solve for w:

$$z = -\frac{1}{10}(w - 3),$$

so

$$-10z = -w + 3,$$

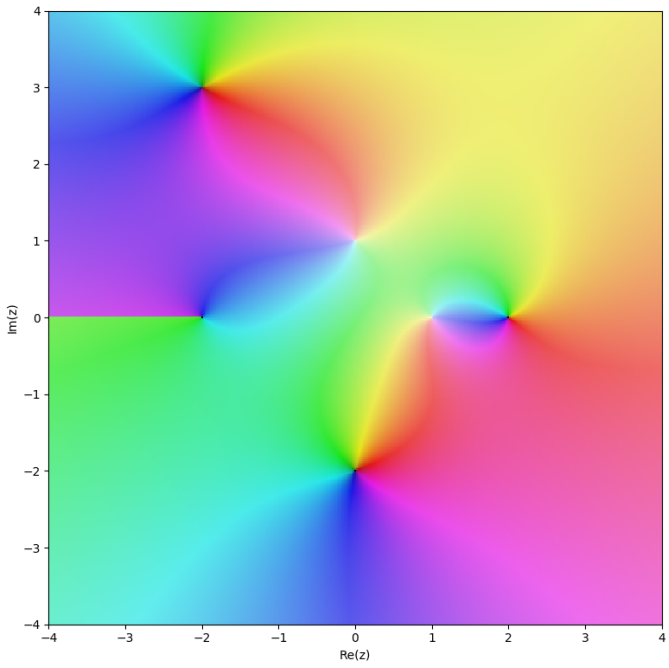
hence

$$w = \underbrace{3 + 10z}_{=f^{-1}(z)}.$$

Therefore,

$$h(z) = f^{-1}(g(z)) = f^{-1}\left(\frac{5}{3}\left(\frac{z-1}{z-3}\right)\right) = 3 + 10\left[\frac{5}{3}\left(\frac{z-1}{z-3}\right)\right]$$

4. (7 points) Identify any zeros, "blow up points", and/or branch cuts appearing in the following domain coloring:

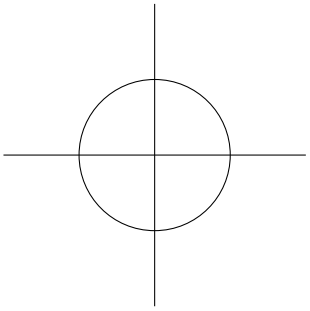


zeros: $-2, -2i, -2 + 3i, 2$ blow-up points: $i, 1$ branch cut: $(-\infty, -2)$

5. (30 points) Draw the contour and compute the integral.

(a) (15 points) $\displaystyle\int_C \frac{1}{|z|}\mathrm{d}z, \quad C: \begin{cases} z(t) = e^{it} \\ 0 \le t \le 2\pi \end{cases}$

Solution:

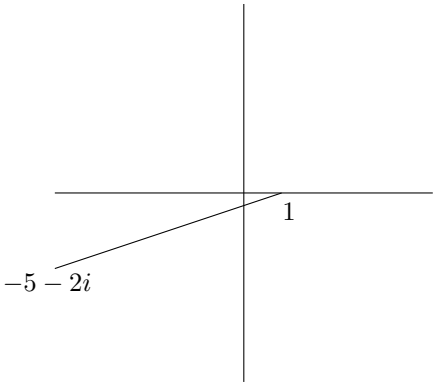


Since $z'(t) = ie^{it}$, we compute

$$\begin{aligned} \int_C \frac{1}{|z|}\mathrm{d}z &= \int_0^{2\pi} \underbrace{\frac{1}{|e^{it}|}}_{=1} ie^{it}\mathrm{d}t \\ &= i\int_0^{2\pi} e^{it}\mathrm{d}t \\ &= i\left[\frac{1}{i}e^{it}\right|_0^{2\pi} \\ &= e^{2\pi i} - e^0 \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

(b) (15 points) $\displaystyle\int_C \mathrm{Im}(z)\mathrm{d}z, \quad C: \begin{cases} z(t) = (-2-i) + (3+i)t \\ -1 \le t \le 1 \end{cases}$

Solution:



Since $z'(t) = 3 + i$ and $\mathrm{Im}(z(t)) = -1 + t$, we compute

$$\begin{aligned} \int_C \mathrm{Im}(z)\mathrm{d}z &= \int_{-1}^{1} (-1+t)(3+i)\mathrm{d}t \\ &= (3+i)\left[-t + \frac{t^2}{2}\right|_{-1}^{1} \\ &= (3+i)\left[\left(-1 + \frac{1^2}{2}\right) - \left(-(-1) + \frac{(-1)^2}{2}\right)\right] \\ &= (3+i)\left(\frac{1}{2} - \frac{3}{2}\right) \\ &= -(3+i) \end{aligned}$$

6. (5 points) (**BONUS**) Use the definitions $\sin(z) = \dfrac{e^z - e^{-iz}}{2i}$ and $\cos(z) = \dfrac{e^z + e^{-z}}{2}$ to establish the product-to-sum formula

$$2\sin(z_1)\sin(z_2) = \cos(z_1 - z_2) - \cos(z_1 + z_2).$$

Solution: Calculate

$$\begin{aligned}
2\sin(z_1)\sin(z_2) &= 2\left(\frac{e^{z_1} - e^{-z_1}}{2i}\right)\left(\frac{e^{z_2} - e^{-z_2}}{2i}\right) \\
&= \frac{2}{4i^2}\left(e^{z_1+z_2} - e^{z_1-z_2} - e^{-z_1+z_2} + e^{-z_1-z_2}\right) \\
&= -\frac{1}{2}e^{z_1+z_2} + \frac{1}{2}e^{z_1-z_2} + \frac{1}{2}e^{-(z_1-z_2)} - \frac{1}{2}e^{-(z_1+z_2)} \\
&= \frac{e^{z_1-z_2} - e^{-(z_1-z_2)}}{2} + \frac{-e^{z_1+z_2} + e^{-(z_1+z_2)}}{2} \\
&= \frac{e^{z_1-z_2} - e^{-(z_1-z_2)}}{2} - \frac{e^{z_1+z_2} - e^{-(z_1+z_2)}}{2} \\
&= \cos(z_1 - z_2) - \cos(z_1 + z_2),
\end{aligned}$$

as was to be shown.