SpTp: MATH 1199 - EXAM 1 FALL 2019 SOLUTION

Wednesday, 18 September
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (15 points)

   (a) (3 points) Write the Cauchy-Riemann equations for a function $f = u + iv$.
   Solution: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

   (b) (3 points) Define the notation $\lim_{z \to z_0} f(z) = L$.
   Solution: For every $\epsilon > 0$ there exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z) - L| < \epsilon$.

   (c) (3 points) Define the derivative $f'(z)$ in terms of a limit.
   Solution: $f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$

   (d) (3 points) Define what it means for a function $f \colon U \to \mathbb{C}$ to be continuous at $z_0 \in U$.
   Solution: $\lim_{z \to z_0} f(z) = f(z_0)$

   (e) (3 points) Explain the difference between arg and Arg.
   Solution: $\mathrm{Arg}(z)$ is the angle of the point $z \in \mathbb{C}$ with the restriction that $-\pi < \mathrm{Arg}(z) \leq \pi$ while $\arg(z)$ is multivalued and obeys $\arg(z) = \mathrm{Arg}(z) + 2n\pi$, for $n \in \mathbb{Z}$

2. (15 points) Write in the form $a + bi$ for $a, b \in \mathbb{R}$.

   (a) (3 points) $(3 - i)(2 + i)$
   Solution: Compute

$$(3 - i)(2 + i) = 6 + 3i - 2i - i^2 = 6 + i + 1 = 7 + i.$$

   (b) (3 points) $\dfrac{1}{1 - i}$
   Solution: Compute

$$\frac{1}{1-i} = \frac{1}{1-i}\left(\frac{1+i}{1+i}\right) = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1}{2} + \frac{1}{2}i.$$

   (c) (3 points) $\overline{3 + i} - 2i$
   Solution: Compute

$$\overline{3+i} - 2i = (3 - i) - 2i = 3 - 3i.$$

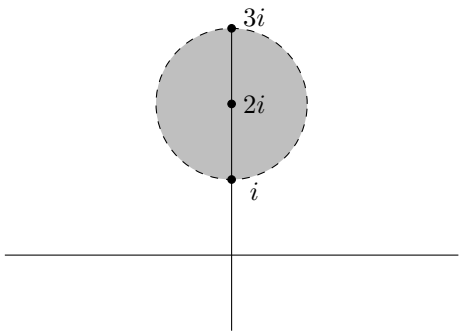
   (d) (3 points) $\mathrm{Log}(-1 + i)$
   Solution: Since $|-1 + i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $\mathrm{Arg}(-1 + i) = \dfrac{3\pi}{4}$, it has polar form $-1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$. Therefore,

$$\mathrm{Log}(-1 + i) = \ln\left(\sqrt{2}\right) + \frac{\pi}{4}i = \frac{1}{2}\ln(2) + \frac{\pi}{4}i.$$

   (e) (3 points) $5e^{-\frac{\pi}{4}i}$
   Solution: Using the Euler formula,

$$5e^{-\frac{\pi}{4}i} = 5\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 5\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \frac{5\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i.$$

3. (6 points) Sketch a picture of the set in the complex plane defined by the inequality $|z - 2i| > 1$.
Solution: This inequality describes the interior of the ball of radius 1 centered at the point $2i$. Therefore we draw it as



4. (9 points) Put the number into polar form $re^{i\theta}$, where $r, \theta \in \mathbb{R}$.
   (a) (3 points) -6
   Solution: Clearly, $|-6| = 6$ and since it lies on the negative real axis, $\text{Arg}(-6) = \pi$. Therefore,
   $$-6 = 6e^{\pi i}.$$
   (b) (3 points) $13i$
   Solution: Clearly $|13i| = 13$ and since it lies on the positive imaginary axis, $\text{Arg}(13i) = \dfrac{\pi}{2}$. Therefore,
   $$13i = 13e^{\frac{\pi}{2}i}.$$
   (c) (3 points) $-1 - i$
   Solution: We observe $|-1-i| = \sqrt{2}$ and since $-1 - i$ is in quadrant IV, an angle for it argument can be found by computing
   $$\pi + \arctan\left(\frac{-1}{-1}\right) = \pi + \arctan(1) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$
   (*note: this angle is not* $\text{Arg}(-1-i)$*, but that's okay for polar form*). Therefore,
   $$-1 - i = \sqrt{2}e^{\frac{5\pi}{4}i}.$$

5. (8 points) Find all 5th roots of $-i$.
Solution: First write $-i$ in polar form: $-i = e^{-\frac{\pi}{2}i}$. Now we may compute the 5th roots of $-i$:
$$(-i)^{\frac{1}{5}} = \left(e^{-\frac{\pi}{2}i}\right)^{\frac{1}{5}} = \exp\left(\left(-\frac{\pi}{10} + \frac{2\pi k}{5}\right)i\right) = \exp\left(\left(\frac{-\pi + 4\pi k}{10}\right)i\right), \quad k \in \mathbb{Z}.$$
To find the roots, just plug in 5 consecutive values of k:

$k =$	$(-i)^{\frac{1}{5}} =$
0	$\exp\left(-\frac{\pi}{10}i\right)$
1	$\exp\left(\frac{3\pi}{10}i\right)$
2	$\exp\left(\frac{7\pi}{10}i\right)$
3	$\exp\left(\frac{11\pi}{10}i\right)$
4	$\exp\left(\frac{15\pi}{10}i\right)$

6. (8 points) Express $\left(\sqrt{3}+i\right)^8$ in the form $a+bi$ for $a,b\in\mathbb{R}$.

Solution: Write $\sqrt{3}+i$ in polar form: first $|\sqrt{3}+i|=\sqrt{\left(\sqrt{3}\right)^2+1^2}=\sqrt{4}=2$ and since it is in quadrant I, its angle can be obtained by $\arctan\left(\dfrac{1}{\sqrt{3}}\right)=\dfrac{\pi}{6}$. Therefore its polar form is $\sqrt{3}+i=2e^{\frac{\pi}{6}i}$. Therefore, we compute

$$\left(\sqrt{3}+i\right)^8=\left(2e^{\frac{\pi}{6}i}\right)^8=2^8e^{\frac{8\pi}{6}i}=2^8e^{\frac{4\pi}{3}i}=2^8\cos\left(\frac{4\pi}{3}\right)+2^8\sin\left(\frac{4\pi}{3}\right)i=-2^7-2^7\sqrt{3}i.$$

7. (5 points) Explain what "$\lim\limits_{z\to\infty}\dfrac{z^2+i}{z-1}=\infty$" means. (*do **not** compute or prove it, just explain the meaning of the notation*)

Solution: It means that

$$\lim_{z\to 0}\frac{1}{\dfrac{\dfrac{1}{z^2}+i}{\dfrac{1}{z}-1}}=0$$

8. (8 points) Write the function $\begin{cases} f\colon\mathbb{C}\to\mathbb{C} \\ f(z)=\dfrac{|z|+1}{\overline{z}} \end{cases}$ in the form $f=u+iv$, where $u,v\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Solution: Let $z=x+iy$ and calculate

$$f(x+iy)=\frac{|x+iy|+1}{\overline{x+iy}}=\frac{\sqrt{x^2+y^2}+1}{x-iy}$$
$$=\frac{\sqrt{x^2+y^2}+1}{x-iy}\left(\frac{x+iy}{x+iy}\right)=\underbrace{\frac{x\sqrt{x^2+y^2}+x}{x^2+y^2}}_{=u(x,y)}=\underbrace{\frac{y\sqrt{x^2+y^2}+y}{x^2+y^2}}_{=v(x,y)}i$$

9. (10 points) Write an ϵ-δ proof that the function $\begin{cases} f\colon\mathbb{C}\to\mathbb{C} \\ f(z)=7z-i \end{cases}$ is continuous.

Proof: Let $\epsilon>0$ and choose $\delta=\dfrac{\epsilon}{7}$. If $|z-z_0|<\delta$, then compute

$$|f(z)-f(z_0)|=|(7z-i)-(7z_0-i)|=|7||z-z_0|<7\delta=7\left(\frac{\epsilon}{7}\right)=\epsilon,$$

completing the proof. ■

10. (8 points) Verify that the Cauchy-Riemann equations hold for the function $\begin{cases} f\colon\mathbb{C}\to\mathbb{C} \\ f(z)=z^2+z \end{cases}$

Solution: First we must write f in the form $f=u+iv$. To do that, let $z=x+iy$ and compute

$$f(x+iy)=(x+iy)^2+(x+iy)=\left(x^2+2xyi+i^2y^2\right)+x+iy=\underbrace{x^2-y^2+x}_{=u(x,y)}+\underbrace{2xy+y}_{=v(x,y)}i.$$

Now compute

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}x^2-y^2+x=2x+1,$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} x^2 - y^2 + x = 2x + 1 = -2y,$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x} 2xy + y = 2y,$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} 2xy + y = 2x + 1.$$

We observe from the calculations that the Cauchy-Riemann equations hold.

11. (8 points) Explain why the function $\begin{cases} f\colon \mathbb{C} \to \mathbb{C} \\ f(z) = \overline{z} - z \end{cases}$ is not complex-differentiable.

Solution: First we will write it in the form $f = u + iv$. Let $z = x + iy$ and compute

$$f(x+iy) = \overline{x+iy} - (x+iy) = x - iy - x - iy = \underbrace{0}_{=u(x,y)} + \underbrace{(-2y)}_{=v(x,y)} i.$$

Now compute

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} 0 = 0,$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y} - 2y = -2.$$

In this case, we see that $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ and by the corollary to the Cauchy-Riemann equations theorem, we conclude that f is not complex-differentiable.