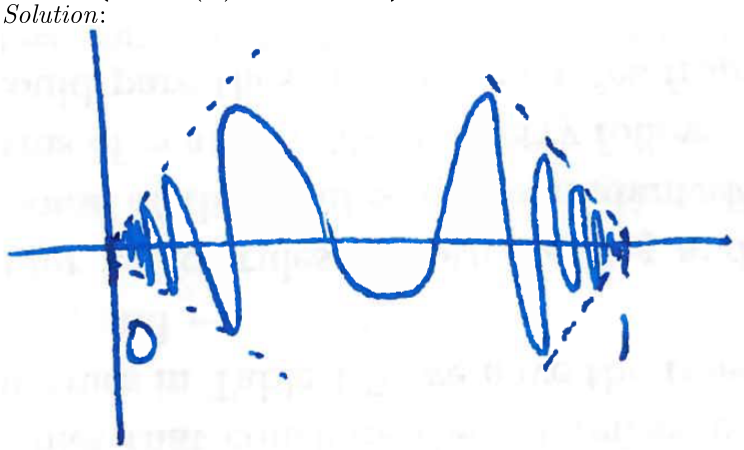
Homework 7 — MATH 4590 Spring 2018

1. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $|f(t) - f(x)| \leq |t - x|^2$ for all $t, x \in \mathbb{R}$. Prove that f is a constant function.
   Proof: Let $\epsilon > 0$ and choose $0 < \delta < \epsilon$. Then rearranging the given inequality implies
   $$\left| \frac{f(t) - f(x)}{t - x} \right| \leq |t - x|.$$
   Taking the limit as $t \to x$ implies
   $$\lim_{t \to x} \left| \frac{f(t) - f(x)}{t - x} \right| \leq \lim_{t \to x} |t - x|,$$
   yielding
   $$|f'(x)| \leq |0|.$$
   Since $|f'(x)| \geq 0$, we must conclude that $f'(x) = 0$, in other words, f is a constant function. ■

2. (a) Draw the graph of a continuous function defined on $[0, 1]$ that is differentiable on $(0, 1)$, but not at the endpoints.
   (*hint: take inspiration from the graph of the following function, which is continuous on* $\mathbb{R}$ *but not differentiable at zero:*
   $$f(x) = \left\{ \begin{array}{ll} 0, & x = 0 \\ x \sin\left(\frac{1}{x}\right), & x \neq 0 \end{array} \right\})$$
   Solution:



   (b) Can you find a formula for such a function? (not necessarily the one you drew)

   (c) Does the Mean Value Theorem apply to such a function? Why or why not?
   Solution: It does apply. MVT would only require differentiability in $(0, 1)$ and continuity on $[0, 1]$, which is satisfied by such a function.

3. Assume that the functions f and g are smooth (i.e. infinitely differentiable). Prove the Leibniz product rule: for any $r \in \{1, 2, 3, \ldots\}$,

$$(f \cdot g)^{(r)}(x) = \sum_{k=0}^{r} \binom{r}{k} f^{(k)}(x) g^{r-k}(x),$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
(*hint: use induction*)
Proof: The case $r = 0$ is true since it says

$$(f \cdot g)^{(r)}(x) = \sum_{k=0}^{0} \binom{r}{k} f^{(k)}(x) g^{(r-k)}(x),$$

i.e.

$$(f \cdot g)(x) = f(x)g(x).$$

Now assume the formula holds for $r = N$. Now compute

$$\begin{aligned}
(f \cdot g)^{(N+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f \cdot g)^{(N)}(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{N} \binom{N}{k} f^{(k)}(x) g^{(N-k)}(x) \\
&\overset{\text{product rule}}{=} \sum_{k=0}^{N} \binom{N}{k} \left[f^{(k+1)}(x) g^{(N-k)}(x) + f^{(k)}(x) g^{(N-k+1)}(x) \right] \\
&= \left(\sum_{k=0}^{N} \binom{N}{k} f^{(k+1)}(x) g^{(N-k)}(x) \right) + \left(\sum_{k=0}^{N} \binom{N}{k} f^{(k)}(x) g^{(N-k+1)}(x) \right) \\
&\overset{\text{reindex}}{=} \left(\sum_{k=1}^{N+1} \binom{N}{k-1} f^{(k)}(x) g^{(N-k+1)}(x) \right) + \left(\sum_{k=0}^{N} \binom{N}{k} f^{(k)}(x) g^{(N-k+1)}(x) \right) \\
&= \underbrace{\binom{N}{N} f^{(N+1)}(x) g^{(0)}(x)}_{k=N+1 \text{ term}} + \left(\sum_{k=1}^{N} \binom{N}{k-1} f^{(k)}(x) g^{(N-k+1)}(x) \right) \\
&\quad + \underbrace{\binom{N}{0} f^{(0)}(x) g^{(N+1)}}_{k=0 \text{ term}} + \left(\sum_{k=1}^{N} \binom{N}{k} f^{(k)}(x) g^{(N-k+1)}(x) \right).
\end{aligned}$$

Notice here that $\binom{N}{N} = 1$ and $\binom{N}{0} = 1$. Combining the sums shows

$$(*) \quad (f \cdot g)^{(N+1)}(x) = f^{(0)}(x) g^{(N+1)} + \left(\sum_{k=1}^{N} \left[\binom{N}{k-1} + \binom{N}{k} \right] f^{(k)}(x) g^{(N-k+1)}(x) \right) + f^{(N+1)}(x) g^{(0)}(x)$$

Now calculate

$$\begin{aligned}
\binom{N}{k-1} + \binom{N}{k} &= \frac{N!}{(k-1)!(N-k+1)!} + \frac{N!}{k!(N-k)!} \\
&= \frac{N!}{(k-1)!(N-k)!}\left(\frac{1}{N-k+1} + \frac{1}{k}\right) \\
&= \frac{N!}{(k-1)!(N-k)!}\left(\frac{N+1}{k(N-k+1)}\right) \\
&= \frac{(N+1)!}{k!(N-k+1)!} \\
&= \binom{N+1}{k}
\end{aligned}$$

This shows that $(*)$ is actually

$$(f \cdot g)^{(N+1)}(x) = f^{(0)}(x)g^{(N+1)}(x) + \left(\sum_{k=1}^{N} \binom{N+1}{k} f^{(k)}(x)g^{(N-k+1)}(x)\right) + f^{(N+1)}(x)g^{(0)}(x),$$

and so since $\binom{N+1}{0} = 1 = \binom{N+1}{N+1}$ we may write $(*)$ as

$$(f \cdot g)^{N+1}(x) = \sum_{k=1}^{N+1} \binom{N+1}{k} f^{(k)}(x)g^{(N-k+1)}(x),$$

completing the proof. ■

4. Recall the rth order Taylor polynomial of a r-times differentiable function $f(x)$ is given by

$$P(h) = \sum_{k=0}^{r} \frac{f^{(k)}(x)}{k!} h^k.$$

Also recall that the remainder $R(h)$ is given by

$$R(h) = f(x+h) - P(h),$$

and (assuing f is $r+1$-times differentiable) that the Taylor Approximation Theorem (part c) says that there is some $\theta \in (0, h)$ so that

$$R(h) = \frac{f^{(r+1)}(\theta)}{(r+1)!} h^{r+1}.$$

We will investigate the following question: what is the max possible error that a 5th order Taylor polynomial for the function $f(x) = \sin(x)$ centered at zero may have in approximating $\sin\left(\frac{1}{2}\right)$?

(a) Find the 5th order Taylor polynomial of $f(x) = \sin(x)$.

Solution: Calculate $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(4)}(x) = \sin(x)$, and $f^{(5)}(x) = \cos(x)$. Therefore the 5th order Taylor polynomial is

$$\begin{aligned} P(h) &= \sum_{k=0}^{5} \frac{f^{(k)}(x)}{k!} h^k \\ &= \sin(x) + \cos(x)h - \frac{\sin(x)}{2}h^2 - \frac{\cos(x)}{6}h^3 + \frac{\sin(x)}{24}h^4 + \frac{\cos(x)}{120}h^5. \end{aligned}$$

(b) Set $x = 0$ in the resulting formula from above (this "centers" the Taylor polynomial near x).
Solution: At $x = 0$ we get

$$P(h) = 0 + h - 0 - \frac{h^3}{6} + 0 + \frac{h^5}{120} = h - \frac{h^3}{6} + \frac{h^5}{120}.$$

(c) Use the Taylor Approximation Theorem (part c) to write a formula (in terms of θ) for $R\left(\frac{1}{2}\right)$ (with $x = 0$).
Solution: Calculate

$$R(h) = \frac{f^{(6)}(\theta)}{6!} h^{r+1},$$

and since $f^{(6)}(x) = -\sin(x)$ this turns into

$$R\left(\frac{1}{2}\right) = \frac{-\sin(x)}{6!}\left(\frac{1}{2}\right)^6.$$

(d) Find a number $\xi \in \mathbb{R}$ such that $\left|R\left(\frac{1}{2}\right)\right| < \xi$, where ξ does not depend on θ (with $x = 0$).
Solution: Since $|-\sin(x)| \le 1$ for all x, we can conclude

$$\left|R\left(\frac{1}{2}\right)\right| \le \frac{1}{2^6 6!} \approx 0.0000217. \blacksquare$$

5. Define $f(x) = \begin{cases} x^2, & x < 0 \\ x + x^2, & x \ge 0. \end{cases}$
Differentiation gives $f''(x) = 2$. This is bogus. Why? (*hint: draw a picture of f and its derivative*)
Solution: The first derivative of f is

$$f'(x) = \begin{cases} 2x, & x < 0 \\ 1 + 2x, & x \ge 0. \end{cases}$$

But we see that the function $f'(x)$ is not continuous at $x = 0$ — from the left of 0 it approaches the height of 0 but from the right it approaches the height 1. Since any differentiable function is continuous, we may conclude that f' is not differentiable since it is not continuous. $\blacksquare$