MATH 3503 - EXAM 3 - FALL 2018
SOLUTION

26 October 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (12 points) Consider the function $f(x,y) = x^2 + e^{xy}$ with $x = 2t + s$ and $y = e^s$. Use **the chain rule** to compute the derivative $\dfrac{\partial f}{\partial s}$.
Solution: Calculate

$$\begin{aligned}\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x + ye^{xy})(1) + (xe^{xy})(e^s) \\ &= \left(4t + 2s + e^s\left(e^{(2t+s)e^s}\right)\right) + \left((2t+s)e^{(2t+s)e^s}e^s\right)\end{aligned}$$

2. (20 points) Consider the function $f(x,y) = \sin\left(x^2 y\right)$.
   (a) (5 points) Compute the gradient of f.
   Solution: Calculate

$$\nabla f = \left\langle \frac{\partial}{\partial x}\sin(x^2y), \frac{\partial}{\partial y}\sin(x^2y)\right\rangle = \left\langle 2xy\cos(x^2y), x^2\cos(x^2y)\right\rangle$$

   (b) (5 points) Compute the directional derivative of f at the point $(1,2)$ in the direction of the vector $\vec{v} = \langle 1, 1\rangle$.
   Solution: First normalize $\vec{v}$ to get

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,1\rangle}{\sqrt{1+1}} = \left\langle \frac{2}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right\rangle.$$

   Now we may compute the directional derivative:

$$\begin{aligned}D_{\vec{u}}f(1,2) &= \nabla f(1,2)\cdot\vec{u} \\ &= \left\langle 4\cos(2), \cos(2)\right\rangle \cdot \left\langle \frac{2}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right\rangle \\ &= \frac{8\cos(2)}{\sqrt{2}} + \frac{3\cos(2)}{\sqrt{2}} \\ &= \frac{11\cos(2)}{\sqrt{2}} \approx -3.23\end{aligned}$$

   (c) (5 points) In what direction does f increase the fastest from the point $(1,2)$?
   Solution: It increases the fastest in the direction of the gradient, i.e. $\nabla f(1,2) = \langle 4\cos(2), \cos(2)\rangle$.

   (d) (5 points) What is the maximum rate of change of f at the point $(1,2)$?
   Solution: The maximum rate of change is given by the magnitude of the gradient, i.e.

$$\left\|\nabla f(1,2)\right\| = \left\|\left\langle 4\cos(2), \cos(2)\right\rangle\right\| = 16\cos^2(2) + \cos^2(2) = 17\cos^2(2) \approx 2.944$$

3. (16 points) Consider the function $f(x,y) = x^3 + 6xy + 3y^2 - 9x$.

   (a) (8 points) Compute f_x and f_y.
   Solution: Compute

$$f_x = 3x^2 + 6y - 9,$$

   and

$$f_y = 6x + 6y.$$

(b) (8 points) Use your answer to (a) to find the critical points of f.
Solution: Find the critical points by finding all solutions of the system of equations

$$f_x = 3x^2 + 6y - 9 \overset{\text{SET}}{=} 0 \quad (i)$$

$$f_y = 6x + 6y \overset{\text{SET}}{=} 0 \quad (ii)$$

From (ii) we see that $x = -y$. Plugging this into (i) yields

$$3(-y)^2 + 6y - 9 = 0,$$

i.e.

$$3y^2 + 6y - 9 = 0.$$

Dividing by 3 yields

$$y^2 + 2y - 3 = 0$$

Solving the quadratic yields solutions $y = -3$ and $y = 1$. Therefore using the fact that $x = -y$ we see that the pairs of critical points we have are $(x, y) = (3, -3)$ and $(x, y) = (-1, 1)$.

4. (16 points) The function $f(x, y) = 3x^2 + 6xy + 2y^3 + 12x - 24y$ has critical points at $(0, -2)$ and $(-5, 3)$ (**you do NOT need to verify this**).

(a) (8 points) Write down D.
Solution: First calculate

$$f_x = 6x + 6y + 12,$$

$$f_y = 6x + 6y^2 - 24,$$

$$f_{xx} = 6,$$

$$f_{yy} = 12y,$$

and

$$f_{xy} = 6.$$

Therefore

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 72y - 36.$$

(b) (8 points) Use the D you found in (a) to classify each critical point as the location of a local minimum, local maximum, a saddle point, or state that there is not enough information to classify it ("inconclusive").
Solution:
Analyze $(0, -2)$
Calculate

$$D(0, -2) = 72(-2) - 36 = -288 - 36 < 0,$$

hence there is a saddle point at $(x, y) = (0, -2)$.
Analyze $(-5, 3)$
Calculate

$$D(-5, 3) = 72(3) - 36 = 216 - 36 > 0,$$

and

$$f_{xx}(-5, 3) = 6 > 0,$$

hence there is a local minimum at $(x, y) = (-5, 3)$.

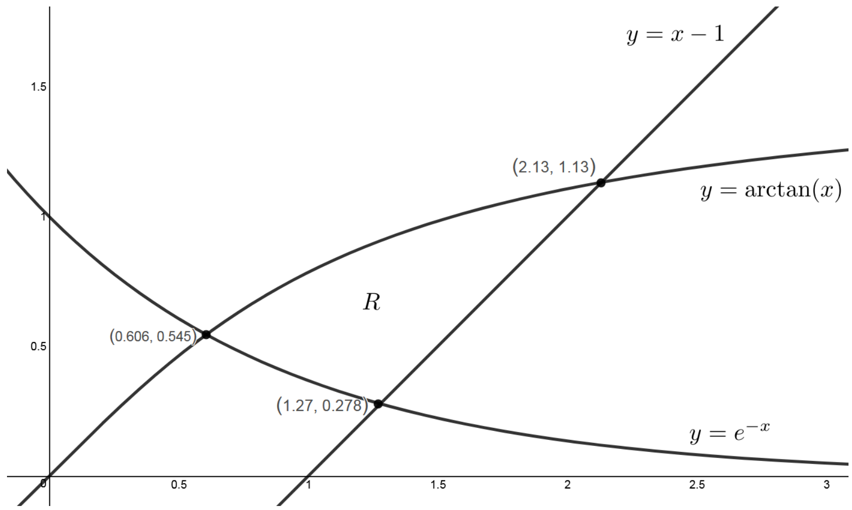
5. Compute the following double integral: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{x}^{\sqrt{\sin(x)}} y \mathrm{d}y \mathrm{d}x.$

Solution: Calculate

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{x}^{\sqrt{\sin(x)}} y \mathrm{d}y \mathrm{d}x &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \underbrace{(\sqrt{\sin(x)})^2}_{=\sin(x)} - x^2 \mathrm{d}x \\
&= \frac{1}{2} \left[-\cos(x) - \frac{x^3}{3} \right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{2} \left[\left(-\cos\left(\frac{\pi}{4}\right) - \frac{1}{3}\left(\frac{\pi}{4}\right)^3 \right) - \left(-\cos\left(-\frac{\pi}{4}\right) - \frac{1}{3}\left(-\frac{\pi}{4}\right)^3 \right) \right] \\
&= \frac{1}{2} \left[-\frac{\sqrt{2}}{2} - \frac{\pi^3}{3(4^3)} + \frac{\sqrt{2}}{2} - \frac{\pi^3}{3(4^3)} \right] \\
&= -\frac{2\pi^3}{6(4^3)} \\
&= -\frac{\pi^3}{3(4^3)} \approx -0.1614
\end{aligned}$$

Do **ONE** of the following two problems. Cross out the one you do NOT want graded.

6. Consider $\iint_R f(x,y)\mathrm{d}A$, where R is the region bounded by curves as in the following picture:



(a) Set up, **but do not evaluate** the double integral as $\mathrm{d}y\mathrm{d}x$.
Solution: "Shooting arrows" in the y direction yields two integrals:

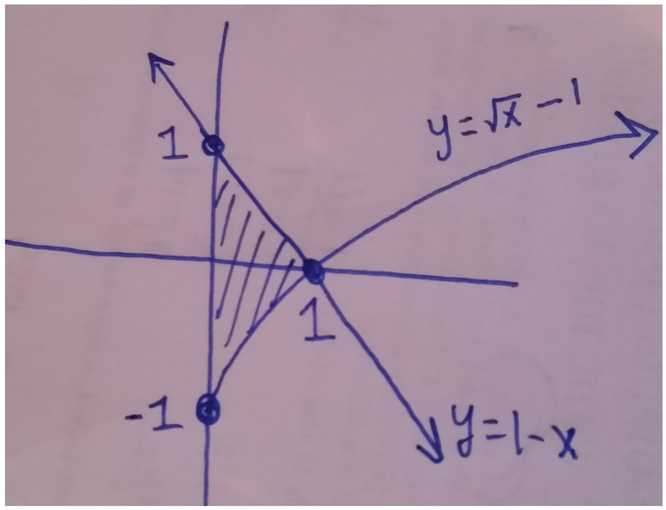
$$\iint_R f(x,y)\mathrm{d}A = \int_{0.606}^{1.27}\int_{e^{-x}}^{\arctan(x)} f(x,y)\mathrm{d}y\mathrm{d}x + \int_{1.27}^{2.13}\int_{x-1}^{\arctan(x)} f(x,y)\mathrm{d}y\mathrm{d}x$$

(b) Set up, **but do not evaluate** the double integral as $\mathrm{d}x\mathrm{d}y$.
Solution: "Shooting arrows" in the x direction yields two integrals:

$$\iint_R f(x,y)\mathrm{d}A = \int_{0.278}^{0.545}\int_{-\ln(y)}^{y+1} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{0.545}^{1.13}\int_{\tan(y)}^{y+1} f(x,y)\mathrm{d}x\mathrm{d}y$$

7. Consider the following double integral: $\int_0^1 \int_{\sqrt{x}-1}^{1-x} f(x,y)\mathrm{d}y\mathrm{d}x$.

(a) Draw the region being graphed. For full credit, label all bounding curves and intersection points between those curves.
Solution:



(b) Set up, **but do not evaluate** the double integral as a $\mathrm{d}x\mathrm{d}y$ double integral.
Solution: Note that $y = \sqrt{x} - 1$ yields $x = (y+1)^2$ and $y = 1 - x$ yields $x = 1 - y$. Now we see

$$\int_0^1 \int_{\sqrt{x}-1}^{1-x} f(x,y)\mathrm{d}y\mathrm{d}x = \int_{-1}^{0} \int_0^{(y+1)^2} \mathrm{d}x\mathrm{d}y + \int_0^1 \int_0^{1-y} \mathrm{d}x\mathrm{d}y$$