MATH 3503 - EXAM 1 - FALL 2018
SOLUTION

7 September 2018
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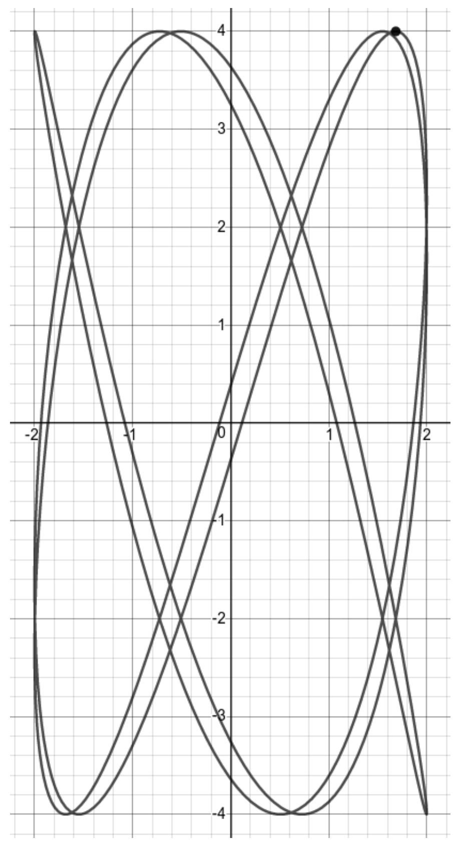
Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (7 points) The curve given by the following parametric equation is called a Lissajous curve:

$$\begin{cases} x = 2\sin(3t+1) \\ \\ y = 4\cos(5t) \end{cases}$$

We may draw it; the point corresponding to $t = 0$ is indicated with a filled black circle at the top right of the plot:



Find an **equation** for the tangent line to this curve at the point corresponding to $t = 0$.
Solution (**this problem resembles #66 from HW1**): First find the point corresponding to parameter value $t = 0$ by plugging $t = 0$ into the parametric equations:

$$x(0) = 2\sin(3(0)+1) = 2\sin(1),$$

and

$$y(0) = 4\cos(5(0)) = 4\underbrace{\cos(0)}_{=1} = 4.$$

Thus we see that the coordinates of the point in question are $(x_0, y_0) = (2\sin(1), 4)$. Calculate

$$x'(t) = 6(1-\cos(t)),$$

and

$$y'(t) = 6\sin(t).$$

Now calculate the slope of the tangent line at any parameter value t:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)} = \frac{6\sin(t)}{6(1-\cos(t))}.$$

Therefore the slope of the tangent line at the point of the graph corresponding to parameter $t = 0$ is

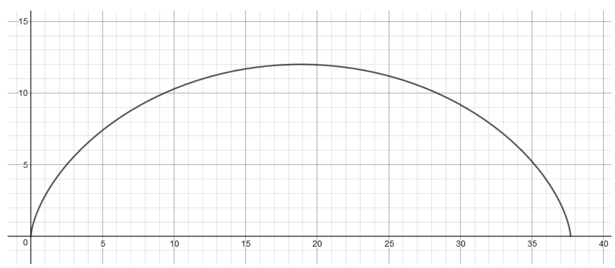
$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=0} = \frac{6\sin(0)}{6(1-\cos(0))} = \frac{0}{6} = 0.$$

Therefore using the point-slope form of the equation of the line, we get the equation to the tangent line to be the line going through $(x_0, y_0) = (2\sin(1), 4)$ with slope $m = 0$, i.e.

$$y - 4 = 0,$$

which is the horizontal line $y = 4$.

2. (8 points) Find the area under the cycloid given by $\begin{cases} x = 6(\theta - \sin(\theta)) \\ y = 6(1-\cos(\theta)) \\ 0 \le \theta \le 2\pi \end{cases}$



Solution (**this problem resembles #102 in HW1**): First calculate

$$x'(\theta) = 6(1-\cos(\theta)).$$

Now, using the formula for area, we compute

$$(*) \qquad \begin{aligned} \text{Area} &= \left|\int_0^{2\pi} y(t)x'(t)\mathrm{d}\theta\right| \\ &= \left|\int_0^{2\pi} 6(1-\cos(\theta))6(1-\cos(\theta))\mathrm{d}\theta\right| \\ &= 36\left|\int_0^{2\pi} 1 - 2\cos(\theta) + \cos^2(\theta)\mathrm{d}\theta\right| \\ &= 36\left[\left|\int_0^{2\pi} 1\mathrm{d}\theta - 2\int_0^{2\pi}\cos(\theta)\mathrm{d}\theta + \int_0^{2\pi}\cos^2(\theta)\mathrm{d}\theta\right|\right] \end{aligned}$$

Since

$$\int_0^{2\pi} 1\mathrm{d}\theta = \theta\Big|_0^{2\pi} = 2\pi - 0 = 2\pi,$$

$$-2\int_0^{2\pi} \cos(\theta)\mathrm{d}\theta = -2\sin(\theta)\Big|_0^{2\pi} = -2\sin(2\pi) - (-2)\sin(0) = 0 - 0 = 0,$$

and

$$\begin{aligned}\int_0^{2\pi} \cos^2(\theta)\mathrm{d}\theta &\overset{\text{trig identity}}{=} \int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{1}{2}\mathrm{d}\theta + \frac{1}{2}\int_0^{2\pi} \cos(2\theta)\mathrm{d}\theta \\ &= \frac{1}{2}(2\pi - 0) + \frac{1}{2}\left[\frac{\sin(2\theta)}{2}\Big|_0^{2\pi}\right. \\ &= \pi + 0 \\ &= \pi,\end{aligned}$$

we now see from $(*)$ that

$$\text{Area} = 36[2\pi - 0 + \pi = 36(3\pi) = 108\pi.$$

3. (8 points) Find the arc length of the parametric curve $\begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = \dfrac{t}{2} \\ 1 \le t \le 5 \end{cases}$

Solution (**this problem resembles #116 in HW1**): First calculate

$$x'(t) = -\sin(t),$$

$$y'(t) = \cos(t),$$

and

$$z'(t) = \frac{1}{2}.$$

Now using the arclength formula, we calculate

$$\begin{aligned}
\text{ArcLength} &= \int_1^5 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\mathrm{d}t \\
&= \int_1^5 \sqrt{(-\sin(t))^2 + (\cos(t))^2 + \left(\frac{1}{2}\right)^2}\mathrm{d}t \\
&\overset{\cos^2(t)+\sin^2(t)=1}{=} \int_1^5 \sqrt{1 + \frac{1}{4}}\mathrm{d}t \\
&= \sqrt{\frac{5}{4}} \int_1^5 1\mathrm{d}t \\
&= \sqrt{\frac{5}{4}} t\Big|_{t=1}^{t=5} \\
&= \sqrt{\frac{5}{4}}(5-1) \\
&= 4\sqrt{\frac{5}{4}} \\
&= 2\sqrt{5}.
\end{aligned}$$

4. (18 points) Consider the vectors $\vec{a} = \langle 1, 4, -1\rangle$ and $\vec{b} = \langle -1, -2, 3\rangle$.

   (a) (6 points) Compute $2\vec{a} - 3\vec{b}$.

   Solution (**this resembles Quiz 1 (i)**): Calculate

$$\begin{aligned}
2\vec{a} - 3\vec{b} &= 2\langle 1, 4, -1\rangle - 3\langle -1, -2, 3\rangle \\
&= \langle 2, 8, -2\rangle - \langle -3, -6, 9\rangle \\
&= \langle 2+3, 8+6, -2-9\rangle \\
&= \langle 5, 14, -11\rangle.
\end{aligned}$$

(b) (6 points) Compute $\left\|\vec{a}+\vec{b}\right\|$.
Solution (**this resembles Quiz 1 (ii)**): Calculate

$$\begin{aligned}\left\|\vec{a}+\vec{b}\right\| &= \|\langle 1,4,-1\rangle + \langle -1,-2,3\rangle\| \\ &= \|\langle 1-1, 4-2, -1+3\rangle\| \\ &= \|\langle 0,2,2\rangle\| \\ &= \sqrt{0^2+2^2+2^2} \\ &= \sqrt{8}.\end{aligned}$$

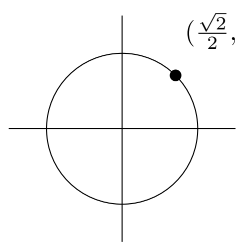
(c) (6 points) Find the unit vector $\vec{u}$ pointing in the same direction as $\vec{a}$.
Solution (**this resembles #10 in HW2**): We know that $\vec{a}$ is not a unit vector because

$$\|\vec{a}\| = \|\langle 1,4,-1\rangle\| = \sqrt{1^2+4^2+(-1)^2} = \sqrt{1+16+1} = \sqrt{18} \underbrace{\neq 1}_{\text{not a unit vector}}.$$

Therefore we must scale $\vec{a}$ to get a unit vector pointing in the same direction as $\vec{a}$ as follows:

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{18}}\langle 1,4,-1\rangle = \left\langle \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right\rangle.$$

5. (7 points) Find the vector $\vec{v}$ in the plane with magnitude $\|\vec{v}\| = 25$ that makes an angle of $\frac{\pi}{4}$ radians with the positive x-axis.
Solution (**this resembles #34 in HW2**): Consider the following portion of the unit circle:



So we observe that a unit vector (it is length 1 because it comes from a point on the **unit** circle) in the desired direction is $\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$. Therefore the desired vector is

$$\vec{v} = 25\vec{u} = 25\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle = \left\langle 25\sqrt{2}, 25\sqrt{2}\right\rangle.$$

6. (14 points) Let $\vec{u} = \langle 1, 3, -1\rangle$ and let $\vec{v} = \langle 5, -1, -2\rangle$.

   (a) (7 points) Compute $\text{proj}_{\vec{u}}\vec{v}$.
   Solution (**this problem resembles #170 a) in HW2**): Using the definiton of proj, compute

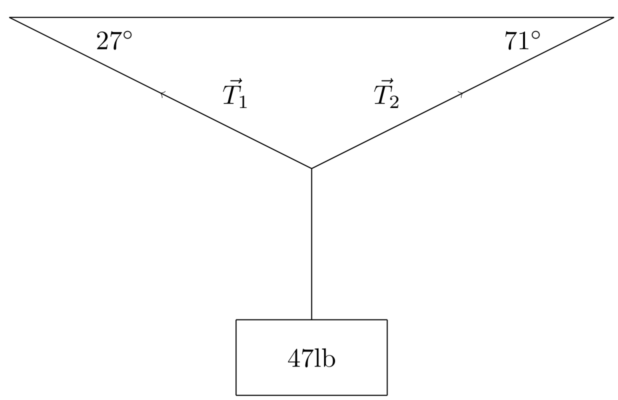
$$\begin{aligned}\text{proj}_{\vec{u}}\vec{v} &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}\\ &= \frac{\langle 1,3,-1\rangle\cdot\langle 5,-1,-2\rangle}{\|\langle 1,3,-1\rangle\|^2}\langle 1,3,-1\rangle\\ &= \frac{5-3+2}{\sqrt{1+9+1}^2}\langle 1,3,-1\rangle\\ &= \frac{4}{11}\langle 1,3,-1\rangle\\ &= \left\langle \frac{4}{11}, \frac{12}{11}, -\frac{4}{11}\right\rangle.\end{aligned}$$

   (b) (7 points) Use your answer from above to compute $\text{comp}_{\vec{u}}\vec{v}$.
   Solution (**this problem resembles #170 b) in HW2**): Using the definition of comp, compute

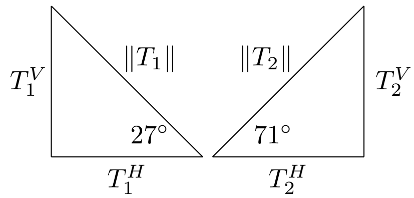
$$\text{comp}_{\vec{u}}\vec{v} = \|\text{proj}_{\vec{u}}\vec{v}\| = \left\|\left\langle \frac{4}{\sqrt{11}}, \frac{12}{\sqrt{11}}, -\frac{4}{\sqrt{11}}\right\rangle\right\| = \sqrt{\frac{16}{121}+\frac{144}{121}+\frac{16}{121}} = \sqrt{\frac{176}{121}} = \frac{\sqrt{176}}{11}.$$

7. (18 points) A 47 lb weight hangs from a rope that makes the angles of 27° and 71° respectively with the horizontal.

   (a) (5 points) Draw a picture of this scenario, letting $\vec{T_1}$ and $\vec{T_2}$ be the force of tension in the cables.
   Solution (**this resembles # 56 on HW2**): Draw the scenario:



   (b) (13 points) Find the **magnitudes** of the forces of tension $\vec{T_1}$ and $\vec{T_2}$ in the cables if the resultant force of the object is zero.
   Solution: First draw the two natural triangles in this problem:



From this we observe:

$$T_1^H = \|T_1\|\cos(27°),$$
$$T_1^V = \|T_1\|\sin(27°),$$
$$T_2^H = \|T_2\|\cos(27°),$$

and

$$T_2^V = \|T_2\|\sin(27°).$$

Since "result force of the object is zero", we arrive at the equations

$$\begin{cases} T_1^H = T_2^H \\ \\ T_1^V + T_2^V = 47 \end{cases}$$

Replacing these with the equivalent expressions involing $\|T_1\|$ and $\|T_2\|$, we obtain

$$\begin{cases} \|T_1\|\cos(27°) = \|T_2\|\cos(71°) & (i) \\ \\ \|T_1\|\sin(27°) + \|T_2\|\sin(71°) = 47 & (ii) \end{cases}$$

From (i) we get

$$\|T_1\| = \|T_2\|\frac{\cos(71°)}{\cos(27°)}.$$

Plugging this into (ii) yields

$$\left(\|T_2\|\frac{\cos(71°)}{\cos(27°)}\right)\sin(27°) + \|T_2\|\sin(71°) = 47.$$

Solving this for $\|T_2\|$ yields

$$\|T_2\| = \frac{47}{\frac{\cos(71°)}{\cos(27°)}\sin(27°) + \sin(71°)},$$

and hence

$$\|T_1\| = \left(\frac{47}{\frac{\cos(71°)}{\cos(27°)}\sin(27°) + \sin(71°)}\right)\frac{\cos(71°)}{\cos(27°)}.$$

8. (6 points) Let $P = (1, 2, 3)$ and let $Q = (5, -1, 2)$. Find an equation for the line containing both P and Q. *Solution* (**this problem resembles HW3 #244 and #248**): The parallel vector is given by

$$\vec{v} = \overrightarrow{PQ} = \langle 5, -1, 2 \rangle - \langle 1, 2, 3 \rangle = \langle 4, -3, -1 \rangle.$$

Therefore, using $(x_0, y_0, z_0) = (1, 2, 3)$ we oberve that the equation of the line is given by the following parametric equations:

$$\begin{cases} x = 1 + 4t \\ y = 2 - 3t \\ z = 3 - t \end{cases}$$

9. (14 points) Let $P = (1, 2, 2)$, let $Q = (2, -1, 4)$, and let $R = (2, 0, 0)$.
(**this problem resembles HW3 #268**)

(a) (4 points) Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
Solution: Calculate

$$\overrightarrow{PQ} = \langle 2, -1, 4 \rangle - \langle 1, 2, 2 \rangle = \langle 1, -3, 2 \rangle$$

and calculate

$$\overrightarrow{PR} = \langle 2, 0, 0 \rangle - \langle 1, 2, 2 \rangle = \langle 1, -2, -2 \rangle$$

(b) (5 points) Find a vector $\vec{n}$ that is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
Solution: Calculate

$$\begin{aligned} \vec{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\ &= \det\left(\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 2 \\ 1 & -2 & -2 \end{bmatrix}\right) \\ &= \langle (-3)(-2) - (2)(-2), -(1(-2) - 2(1)), 1(-2) - (-3)(1) \rangle \\ &= \langle 10, 4, 1 \rangle \end{aligned}$$

(c) (5 points) Use your answer from part b to find an equation of the plane containing P, Q, and R.
Solution: We write the equation of the plane containing the point $(x_0, y_0, z_0) = (1, 2, 2)$ with normal vector $\vec{n} = \langle 10, 4, 1 \rangle$:

$$10(x - 1) + 4(y - 2) + (z - 2) = 0.$$