Homework 8 — MATH 2510 Spring 2018

Consider the vocabulary $\mathcal{V} = \{\preceq\}$. Recall the following $\mathcal{V}$-sentences (called "axioms") from Tuesday's class:

1. **Axiom 1** (reflexive) $\forall x(x \preceq x)$
2. **Axiom 2** (anti-symmetric). $\forall x \forall y(x \preceq y \wedge y \preceq x \rightarrow x = y)$
3. **Axiom 3** (transitive). $\forall x \forall y \forall z(x \preceq y \wedge y \preceq z \rightarrow x \preceq z)$
4. **Axiom 4** (total). $\forall x \forall y(x \preceq y \vee y \preceq x)$

And recall the "well-ordering principle": **every** nonempty subset of the universe contains a least element. (*note: writing this in logic symbols necessitates something like "$\forall S \subseteq \mathcal{U}$" which is not part of our language of first order logic — it is "second order logic"!*)
We say that a structure $M = (S| \preceq)$ is a preorder on S if it is reflexive and transitive. We say that a structure $M = (S| \preceq)$ is a partial order on S if it is an anti-symmetric preorder. We say that a structure $M = (S| \preceq)$ is a total order on S if it is a total partial order. We say that a structure $M = (S| \preceq)$ is a well-order on S if it is a total order that also obeys the well-ordering principle.

Problems

In every problem, determine which of the axioms that the structure models. If the structure is a preorder, partial order, total order, or well-order, then state so.

1. Consider the structure $M = (\{\emptyset, \{a\}, \{b\}, \{a, b\}\}| \preceq)$ where $x \preceq y$ is interpreted to mean $x \subseteq y$.
Solution: Satisfies Axioms 1, 2, and 3. Not Axiom 4 because $\{a\} \not\subset \{b\}$ and $\{b\} \not\subset \{a\}$. It is well-ordered. This structure is a partial order (it is not a well order since it is not a total order). (Such a structure is called well partial order.)

2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function $f(x) = x^2$. Consider the structure $M = (\mathbb{R}| \preceq)$, where $x \preceq y$ is interpreted to mean $f(x) \leq f(y)$ (where "$\leq$" here is interpreted in the usual way).
Solution: Satisfies Axioms 1, 3, and 4. Does not satisfy Axiom 2 since $f(-1) = f(1) = 1$ and therefore $f(-1) \preceq f(1) \wedge f(1) \preceq f(-1)$ while $1 \neq -1$. Does not satisfy WO because the set $(0, 1]$ does not have a $\preceq$-least element. This structure is a preorder. (In fact, a structure that satisfies Axioms 1, 3, and 4 is called a total preorder.)

3. Consider the xy-plane $\mathbb{R}^2 = \{(x, y) \colon x \in \mathbb{R}, y \in \mathbb{R}\}$. Consider the structure $M = (\mathbb{R}| \preceq)$ where $(x_1, y_1) \preceq (x_2, y_2)$ is interpreted to mean that $x_1 \leq x_2$ and $y_1 \leq y_2$ (where "$\leq$" here is interpreted in the usual way).
(*note: this is called the product order of* $\leq$)
Solution: This structure satisfies Axioms 1, 2, and 3. It is not a total order because, for example neither $(1, 0) \preceq (0, 1)$ nor $(0, 1) \preceq (1, 0)$. It is not a well-order because the set $\{(x, y) \colon 0 < x < 1, y = 0\}$ has no least element. This structure is a partial order.

4. Consider the xy-plane $\mathbb{R}^2$. Consider the structure $M = (\mathbb{R}| \preceq)$ where $(x_1, y_1) \preceq (x_2, y_2)$ is interpreted to mean that $x_1 < x_2$ OR $x_1 = x_2$ and $y_1 \leq y_2$ (where "$<$" and "$\leq$" are interpreted in the usual way). So for example, $(-2, 5) \leq (5, 2000)$ because $-2 < 5$ and $(-2, 232) \leq (-2, 333)$ because $-2 = -2$ and $232 \leq 333$.
(*note: this is called "lexicographic" or "dictionary" order*)
Solution: Satisfies Axioms 1, 2, 3, 4. It is not a well order because the set $\{(x, y) \colon 0 < x < 1, y = 0\}$ has no $\preceq$-least element. This structure is a total order.

5. (Sharkovsky order) Consider the structure $M = (\mathbb{N}| \preceq)$, where $\preceq$ is interpreted to be the so-called Sharkovskii order:

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec \; 2^n \; \prec \ldots \quad \cdots \prec \; 7 \cdot 2^n \; \prec \; 5 \cdot 2^n \; \prec \; 3 \cdot 2^n \; \prec \ldots$$
$$\cdots \prec \; 7{\cdot}2 \; \prec \; 5{\cdot}2 \; \prec \; 3{\cdot}2 \; \prec \cdots \prec \; 9 \; \prec \; 7 \; \prec \; 5 \; \prec \; 3 \, .$$

You can understand this interpretation by thinking of it as saying

> descending powers of 2 (ascending) $\preceq \ldots \preceq$ (odd numbers)$\cdot 2^n \preceq$ (odd numbers)$\cdot 2^{n-1} \preceq \ldots \preceq$ (odd numbers)$\cdot 2 \preceq$ odd numbers (descending)

(*note: this order is fundamental to Sharkovskii's theorem from the 1960's*)
Solution: Satisfies Axioms 1, 2, 3, 4. It is not a well-order because the set of odd numbers has no least element. The structure is a total order.

Recall that we defined the notation $A \subseteq B$ to abbreviate the formula $z \in A \to z \in B$. We write $A \cap B$ as and abbreviation for $z \in A \wedge z \in B$.

6. Prove that that $X \cap Y \subseteq X$.
Solution: We need to show that $\forall z((z \in X \cap Y) \to z \in X)$.
General strategy for proving $P \to Q$: Assume P and try to prove Q from it. (This is similar to "$\to$-definition" from our proof system of propositional logic.)
So assume $z \in X \cap Y$. From this we see from the definition of $\cap$ that $z \in X \wedge z \in Y$. Therefore we may conclude that $z \in X$. So, we have shown $z \in X \cap Y \to z \in X$. Since the variable "$z$" did not come from removing an existential quantifier, we may place $\forall z$ in front to get

$$\forall z(z \in X \cap Y \to z \in X),$$

completing the proof. ■