Homework 10 — MATH 2510 Spring 2018

Recall: an ordinal is a transitive well-ordered set. The first few ordinals are: $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{0,1\}, 3 = \{0,1,2\}, 4 = \{0,1,2,3\}, \ldots$. The first infinite ordinal is $\omega = \{0,1,2,3,\ldots\}$. We think of ordinals as being "ordered by ϵ", i.e. if $\alpha \in \beta$, then we think of $\alpha < \beta$. A function $f \colon A \to B$ is called one-to-one provided that whenever $f(x) = f(y)$, it follows that $x = y$. For sets A and B, we say that $|A| \leq |B|$ (note this use of "$\leq$" is different than earlier − this one is for cardinality...if confusing use $\preceq$ for cardinal inequality) provided there is a one-to-one function $f \colon A \to B$. A cardinal number is an ordinal number with the property that if $\beta < \alpha$ (ordinals) then it follows that $|\beta| < |\alpha|$ (cardinals). Let α and β be cardinal numbers, we define cardinal addition by

$$\alpha \oplus \beta = \left|(\alpha \times \{0\}) \bigcup (\beta \times \{1\})\right|.$$

Cardinal multiplication is defined by

$$\alpha \otimes \beta = |\alpha \times \beta|.$$

If A and B are sets, then ${}^B A$ denotes the set of functions whose domain is B and whose codomain is A.

$${}^B A = \left\{ f \colon B \to A \,\middle|\, f \text{ is a function} \right\}.$$

Cardinal exponentiation is defined by

$$\alpha^\beta = |{}^\beta \alpha|.$$

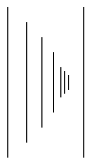
(note: an ordinal exponentiation was computed in HW9, problem 6)
Considered a set A and an order relation $\leq$ on A − we say that a subset $B \subset A$ is cofinal with A if it obeys the following property:

$$\forall a \in A \exists b \in B (a \leq b).$$

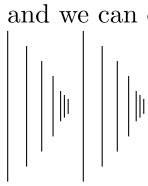
The cofinality of a set A is defined to be the minimum cardinality of all sets which are cofinal with A, i.e.

$$\text{cf}(A) = \min\{|B| \colon B \text{ is cofinal with } A\}.$$

1. Show that the ordinals $|\omega + 2| = |\omega + \omega|$ by demonstrating appropriate one-to-one functions.
   hint: recall that we can draw $\omega + 2$ in the following way:

   and we can draw $\omega + \omega$ in the following way:

2. Consider the set $X = \{0, 1, 2, 3\}$ and the standard order relation $\leq$.

   (a) List all subsets of X (i.e. find $\mathscr{P}(X)$).

   (b) Cross out all subsets of X that are **not cofinal** with X.

   (c) Write down the cardinalities of all cofinal subsets of X.

   (d) Write $\mathrm{cf}(X)$.

   (e) (Bonus): What is $\mathrm{cf}(\omega + 1)$? (*hint*: recall that $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$)

3. Calculate the requested cardinal arithmetic operation. Carefully do it by the definition (i.e. show your sets, etc).

   (a) $2 \oplus 2$

   (b) $2 \otimes 3$

   (c) 3^2