MATH 2510 - EXAM 3 - SPRING 2018
SOLUTION

19 April 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (16 points) Consider the vocabulary $\mathcal{V} = \{\preceq\}$. Recall the following $\mathcal{V}$-sentences (called "order axioms"):

   Axiom 1 (reflexive) $\forall x(x \preceq x)$

   Axiom 2 (anti-symmetric). $\forall x \forall y(x \preceq y \wedge y \preceq x \to x = y)$

   Axiom 3 (transitive). $\forall x \forall y \forall z(x \preceq y \wedge y \preceq z \to x \preceq z)$

   Axiom 4 (total). $\forall x \forall y(x \preceq y \vee y \preceq x)$

   And recall the "well-ordering principle": **every** nonempty subset of the universe contains a least element.

   We say that a structure $M = (S| \preceq)$ is a preorder on S if it is reflexive and transitive. We say that a structure $M = (S| \preceq)$ is a partial order on S if it is an anti-symmetric preorder. We say that a structure $M = (S| \preceq)$ is a total order on S if it is a total partial order. We say that a structure $M = (S| \preceq)$ is a well-order on S if it is a total order that also obeys the well-ordering principle.

   In every problem, determine which of the axioms that the structure models. If the structure is a preorder, partial order, total order, or well-order, then state so.

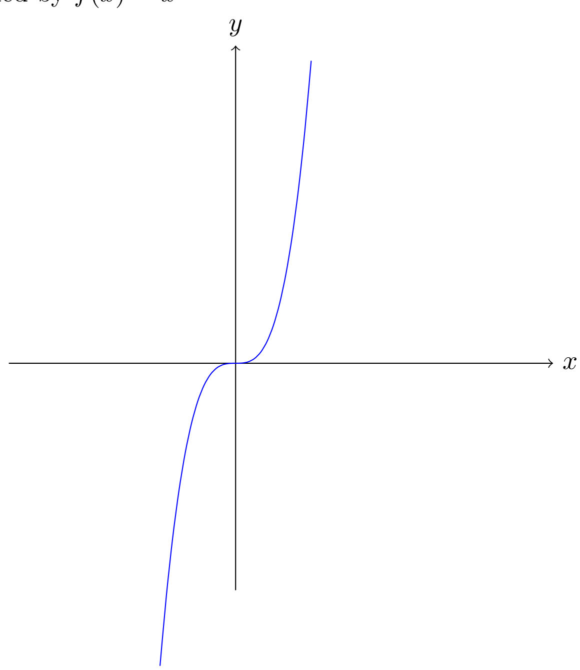
   (a) (5 points) The structure $(\mathbb{R}, \preceq)$ where $x \preceq y$ is interpreted as the "usual" $\leq$ ordering.
   Solution: Axioms 1-4 hold. The well-ordering principle does not hold because, for example, $(0, 1) \subset \mathbb{R}$ does not contain a least element.

   (b) (5 points) The structure $(\mathbb{N}, \preceq)$ where $x \preceq y$ is interpreted as $y \leq x$ in the "usual" ordering (note: this **is not** the usual ordering...read it carefully!).
   Solution: Axioms 1-4 hold, but it is not well-ordered because $\{2, 4, 6, 8, \dots\}$ does not contain a $\preceq$-least element.

   (c) (6 points) Consider $X = \{1, 2, 3, 4, 6, 12, 24\}$. Consider the structure $(X, \preceq)$, where $x \preceq y$ is interpreted to mean x divides evenly into y.
   Solution: Axioms 1-3 hold but Axiom 4 does not hold because $2 \not| 3$ and $3 \not| 2$, meaning $2 \not\preceq 3$ and $3 \not\preceq 2$. It cannot be well-ordered because $\{2, 3\}$ does not contain a $\preceq$-least element.

2. (10 points) Is the function one-to-one? If it is not, then explain why not. If it is, then draw a sketch of the function.

   (a) (5 points) $f \colon \mathbb{Z} \to \mathbb{R}$ defined by $f(x) = x^3$



   Solution: It is one-to-one.

(b) (5 points) $f\colon \{-1,0,1,2\} \to \mathbb{N}$ defined by $f(x) = x^2 + x = 0$
Solution: It is not one-to-one, because $f(-1) = 0 = f(0)$ but $-1 \neq 0$.

3. (16 points) In this problem you will prove $X \cap X = X$.

(a) (5 points) Prove $X \cap X \subseteq X$.
Solution: If $x \in X \cap X$, then $x \in X \wedge x \in X$, hence $x \in X$. Therefore we have shown $x \in X \cap X \to x \in X$, in other words, $X \cap X \subseteq X$.

(b) (5 points) Prove $X \subseteq X \cap X$.
Solution: If $x \in X$ then $x \in X \wedge x \in X$, hence $x \in X \cap X$, i.e. $X \subseteq X \cap X$.

(c) (6 points) Prove that $X \cap X = X$ using an axiom of naive set theory and parts (a) and (b).
Solution: We know from (a) and (b) that $x \in X \to x \in X \cap X$ and that $x \in X \cap X \to x \in X$. Thus we can write $x \in X \leftrightarrow x \in X \cap X$. By the axiom of extensionality, removing the $\forall z$ by replacing z with x, removing the $\forall x$ by replacing x with $X \cap X$, and remoing the $\forall y$ by replacing y with X yields

$$(x \in X \cap X \leftrightarrow x \in X) \to X \cap X = X.$$

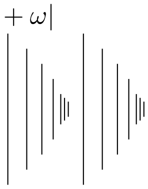
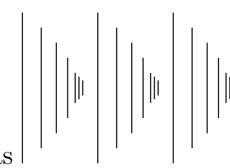
Since we already knew $x \in X \cap X \leftrightarrow x \in X$, we may conclude via $\to$-elimination that $X = X \cap X$.

4. (16 points) Find appropriate one-to-one function(s) that demonstrate the following.

(a) (5 points) $|\mathbb{N}| = |\{2,4,6,8,10,\ldots\}$
Solution: The function $f\colon \mathbb{N} \to \{2,4,6,8,\ldots\}$ defined by $f(n) = 2n$ is one-to-one, showing $|\mathbb{N}| \leq |\{2,4,6,8,\ldots\}|$. Conversely, the function $g\colon \{2,4,6,\ldots\} \to \mathbb{N}$ defined by $g(n) = \dfrac{n}{2}$ is one-to-one and shows that $|\{2,4,6,8,\ldots\}| \leq |\mathbb{N}|$.

(b) (5 points) $|\mathbb{N}| \leq |\mathbb{R}|$
Solution: Define $f\colon \mathbb{N} \to \mathbb{R}$ by $f(n) = n$. This function is one-to-one, proving that $|\mathbb{N}| \leq |\mathbb{R}|$.

(c) (6 points) $|\omega + \omega| = |\omega + \omega + \omega|$

hint: we can draw $\omega + \omega$ as  and we can draw $\omega + \omega + \omega$ as 

Solution: Define $f\colon \omega + \omega \to \omega + \omega + \omega$ by $f(n) = n$. This function is one-to-one, proving that $|\omega + \omega| \leq |\omega + \omega + \omega$. Define $g\colon \omega + \omega + \omega \to \omega + \omega$ for $n \in \mathbb{N}$ by $g(n) = 3n$, for $\omega + n$ by $g(\omega + n) = 3n + 1$ and for $\omega + \omega + n$ by $g(\omega + \omega + n) = 3n + 2$. This maps $\omega + \omega + \omega$ into ω and is one-to-one, proving that $|\omega + \omega + \omega| \leq |\omega + \omega|$. Thus we may conclude that $|\omega + \omega| = |\omega + \omega + \omega|$.

5. (15 points) Consider the set $X = \{2,4,6\}$ with the standard order relation $\leq$.

(a) (5 points) List all subsets of X (i.e. find $\mathscr{P}(X)$).
Solution: $\mathscr{P}(X) = \{\emptyset, \{2\}, \{4\}, \{6\}, \{2,4\}, \{2,6\}, \{4,6\}, \{2,4,6\}\}$.

(b) (5 points) Cross out all subsets of X in part (a) that are **not cofinal** with X.
Solution: $\{\emptyset, \cancel{\{2\}}, \cancel{\{4\}}, \{6\}, \cancel{\{2,4\}}, \{2,6\}, \{4,6\}, \{2,4,6\}\}$

(c) (2 points) Write down the cardinalities of all cofinal subsets of X.
Solution: $|\{6\}| = 1$, $|\{2,6\}| = 2$, $|\{4,6\}| = 2$, and $|\{2,4,6\}| = 3$

(d) (3 points) Write $\mathrm{cf}(X)$.
Solution: $\mathrm{cf}(X) = 1$

6. (15 points) Show full details in computing the following cardinal arithmetic problems. **Not showing all sets (or functions) involved will not yield full credit.**

(a) (4 points) $1 \oplus 2$
Solution: Calculate

$$\begin{aligned} 1 \oplus 2 &= \Big| \{0\} \times \{0\} \bigcup \{0,1\} \times \{1\} \Big| \\ &= \Big| \{(0,0),(0,1),(1,1)\} \Big| \\ &= 3 \end{aligned}$$

(b) (5 points) $3 \otimes 3$
Solution: Calculate

$$\begin{aligned} 3 \otimes 3 &= \Big| \{0,1,2\} \times \{0,1,2\} \Big| \\ &= \Big| \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\} \Big| \\ &= 9 \end{aligned}$$

(c) (6 points) 5^1
Solution: We need to count all functions $f\colon 1 \to 5$, i.e. $f\colon \{0\} \to \{0,1,2,3,4\}$. Since the domain only contains 0, specifying where 0 goes completely defines the function. So we have $f(0) = 0, f(0) = 1, f(0) = 2, f(0) = 3, f(0) = 4$, and $f(0) = 5$ as possible ways to define the function f (not all at the same time!!!!). Therefore there are a total of 5 functions in the set ${}^1 5$. Thus we have

$$5^1 = |{}^1 5| = 5.$$

7. (12 points) Consider a new theory called "differential algebra theory" which has a unary function ∂ and two binary functions $+$ and $\cdot$ along with the following axioms:
Axiom 1: $(\forall x)(\forall y)(\partial(x \cdot y) = (\partial x) \cdot y + x \cdot (\partial y))$
Axiom 2: $(\forall x)(\forall y)(\partial(x + y) = \partial(x) + \partial(y))$
Prove

$$\partial((f \cdot g) \cdot h) = ((\partial f) \cdot g + f \cdot (\partial g)) \cdot h + (f \cdot g) \cdot (\partial h)$$

in differential algebra theory. (*hint: when you "remove $\forall$" in Axiom 1, let $x = f \cdot g$ and let $y = h$*)
Solution: By Axiom 1, remove $\forall x$ by replacing x with $f \cdot g$ and remove $\forall y$ by replacing y with h to get

$$(*) \qquad \partial((f \cdot g) \cdot h) = \Big(\partial(f \cdot g) \cdot h + (f \cdot g) \cdot (\partial h) \Big) + (f \cdot g) \cdot (\partial h).$$

To complete the proof, we must take care of $\partial(f \cdot g)$. To do this, in Axiom 1, remove $\forall x$ by replacing x with f and remove $\forall y$ by replacing y with g to get

$$\partial(f \cdot g) = (\partial f) \cdot g + f \cdot (\partial g).$$

Replacing $\partial(f \cdot g)$ in $(*)$ yields the desired result.

8. (4 points) (**Bonus**):

(a) What is $\mathrm{cf}(\omega + \omega)$? (*hint*: recall that $\omega + \omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$)
Solution: We know $\mathrm{cf}(\omega + \omega) = \omega$ because any cofinal subset of $\omega + \omega$ must be unbounded and $X = \{\omega, \omega + 1, \omega + 2, \ldots\}$ is cofinal (no finite subset of it is confinal!! Why is that?).

(b) Compute the cardinal exponentiation 2^0 and show all the details.
Solution: To compute this, we must count the number of functions in ${}^0 2$, i.e. the number of functions whose domain is $\emptyset$ and whose codomain is $\{0, 1\}$. The only possible function that does this is the "empty function" $\emptyset$ (same as empty set...). Thus $2^0 = |{}^0 2| = 1$.