MATH 1540 - EXAM 3 FALL 2018 SOLUTION

Friday, 16 November 2018
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$$

$$\sin(2\theta)=2\sin(\theta)\cos(\theta)$$

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta)-\sin^2(\theta)\\ &= 1-2\sin^2(\theta)\\ &= 2\cos^2(\theta)-1\end{aligned}$$

$$\sin\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos(\alpha)}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1+\cos(\alpha)}{2}}$$

$$\sin^2(\theta)=\frac{1-\cos(2\theta)}{2}$$

$$\cos^2(\theta)=\frac{1+\cos(2\theta)}{2}$$

Law of sines

$$\frac{\sin(\alpha)}{a}=\frac{\sin(\beta)}{b}=\frac{\sin(\gamma)}{c}$$

Law of cosines

$$a^2=b^2+c^2-2bc\cos(\alpha)$$
$$b^2=a^2+c^2-2ac\cos(\beta)$$
$$c^2=a^2+b^2-2ab\cos(\gamma)$$

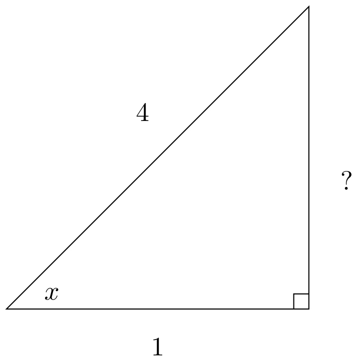
1. (10 points) Find an exact value for $\cos\left(\frac{\pi}{8}\right)$

Solution: Since $\frac{\pi}{4}$ is in QI,

$$\cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{1}{2}\left(\frac{\pi}{4}\right)\right) \overset{\text{half-angle}}{=} +\sqrt{\frac{1+\cos(\frac{\pi}{4})}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

2. (10 points) Given that $\cos(x) = -\frac{1}{4}$ and x is in quadrant III, compute $\sin(2x)$

Solution: First draw a triangle corresponding to $\cos(x) = -\frac{1}{4}$:



Find the missing side using the Pythagorean theorem: $1^2+?^2 = 4^2$, hence $? = \sqrt{15}$. Therefore since x is in QIII, $\sin(x) = -\frac{\sqrt{15}}{4}$. Now we may compute

$$\sin(2x) \overset{\text{double angle}}{=} 2\sin(x)\cos(x) = 2\left(-\frac{\sqrt{15}}{4}\right)\left(-\frac{1}{4}\right) = \frac{\sqrt{15}}{8}.$$

3. (15 points) Prove the identity.

(a) $\dfrac{\sin(x+h)-\sin(x)}{h} = \dfrac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h}$

Solution: Start with the left:

$$\begin{aligned}\frac{\sin(x+h)-\sin(x)}{h} &\overset{\text{sum formula}}{=} \frac{\Big(\sin(x)\cos(h)+\cos(x)\sin(h)\Big)-\sin(x)}{h} \\ &= \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h},\end{aligned}$$

as was to be shown.

(b) $\cos(3x) = \cos(x) - 4\sin^2(x)\cos(x)$

Solution: Notice that $3x = x + 2x$. So, start with the left and compute

$$\begin{aligned}\cos(3x) &= \cos(x+2x) \\ &\overset{\text{sum formula}}{=} \cos(x)\cos(2x) - \sin(x)\sin(2x) \\ &\overset{\text{double angle formulas}}{=} \cos(x)\Big(1-2\sin^2(x)\Big) - \sin(x)\Big(2\sin(x)\cos(x)\Big) \\ &\overset{\text{distribute}}{=} \cos(x) - 2\sin^2(x)\cos(x) - 2\sin^2(x)\cos(x) \\ &= \cos(x) - 4\sin^2(x)\cos(x),\end{aligned}$$

as was to be shown.

4. (15 points) Find an exact value for

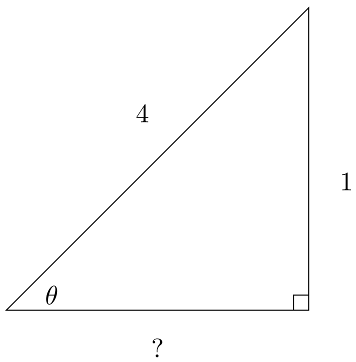
$$\cos\left(\sin^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right).$$

Solution: First use the sum formula to see

$$\begin{aligned}
&\cos\left(\sin^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right) \\
&\quad = \cos\left(\sin^{-1}\left(\frac{1}{4}\right)\right)\cos\left(\tan^{-1}\left(\frac{1}{7}\right)\right) - \sin\left(\sin^{-1}\left(\frac{1}{4}\right)\right)\sin\left(\tan^{-1}\left(\frac{1}{7}\right)\right) \\
&\quad = \cos\left(\sin^{-1}\left(\frac{1}{4}\right)\right)\cos\left(\tan^{-1}\left(\frac{1}{7}\right)\right) - \frac{1}{4}\sin\left(\tan^{-1}\left(\frac{1}{7}\right)\right)
\end{aligned}$$

<u>Deal with $\sin^{-1}$</u>

Let $\theta = \sin^{-1}\left(\frac{1}{4}\right)$ (so θ is in QI or QIV), thus $\sin(\theta) = \frac{1}{4}$ (so θ is in QI or QII). We conclude that θ is in QI. Now draw a triangle:

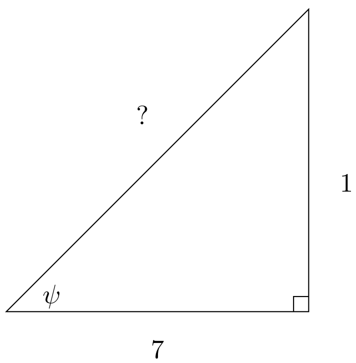


Using Pythagorean theorem to find ? yields $? + 1^2 = 4^1$, i.e. $? = \sqrt{15}$. Therefore we can conclude that

$$\cos\left(\sin^{-1}\left(\frac{1}{4}\right)\right) = \cos(\theta) = \frac{\sqrt{15}}{4}.$$

<u>Deal with $\tan^{-1}$</u>

Let $\psi = \tan^{-1}\left(\frac{1}{7}\right)$ (so ψ is in QI or QIV), thus $\tan(\psi) = \frac{1}{7}$ (so ψ is in QI or QIII). We conclude that ψ is in QIII. Now draw a triangle:



Using Pythagorean theorem to find ? yields $7^1 + 1^2 = ?^2$, i.e. $? = \sqrt{50}$. Therefore we can conclude that

$$\cos\left(\tan^{-1}\left(\frac{1}{7}\right)\right) = \cos(\psi) = \frac{7}{\sqrt{50}},$$

and

$$\sin\left(\tan^{-1}\left(\frac{1}{7}\right)\right) = \sin(\psi) = \frac{1}{\sqrt{50}}.$$

Finally we now see that

$$\begin{aligned}\cos\left(\sin^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right) &= \cos(\theta)\cos(\psi) - \frac{1}{4}\sin(\psi) \\ &= \left(\frac{\sqrt{15}}{4}\right)\left(\frac{7}{\sqrt{50}}\right) - \left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{50}}\right).\end{aligned}$$

5. (10 points) Solve the equation for $0 \leq \theta \leq 2\pi$.

   (a) $\cos(\theta) - \dfrac{1}{2} = 0$
   Solution: First solve for $\cos(\theta)$ to get

   $$\cos(\theta) = \frac{1}{2}.$$

   Now we write down all angles θ in the unit circle whose cosine is $\frac{1}{2}$, i.e. $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

   (b) $2\sin^2(\theta) - \sin(\theta) - 1 = 0$
   Solution: This is a quadratic equation of the form $ax^2+bx+c = 0$ where "x" is actually "$\sin(\theta)$" and $a = 2, b = -1, c = -1$. Therefore we use the quadratic formula to see

   $$\sin(\theta) = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-1)}}{2(2)} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4},$$

   meaning we get two solutions: the "+" solution

   $$\sin(\theta) = \frac{1+3}{4} = \frac{4}{4} = 1$$

   and the "−" solution

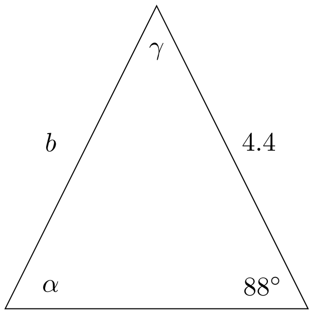
   $$\sin(\theta) = \frac{1-3}{4} = \frac{-2}{4} = -\frac{1}{2}.$$

   Solving $\sin(\theta) = 1$ yields $\theta = \frac{\pi}{2}$ and solving $\sin(\theta) = -\frac{1}{2}$ yields $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

6. (10 points) Let $\vec{u} = \langle 1, 2\rangle$ and let $\vec{v} = \langle -1, 1\rangle$. Compute $2\vec{u} - 3\vec{v}$.
   Solution: Compute

$$\begin{aligned}2\vec{u} - 3\vec{v} &= 2\langle 1, 2\rangle - 3\langle -1, 1\rangle \\ &= \langle 2, 4\rangle + \langle 3, -3\rangle \\ &= \langle 2+3, 4+(-3)\rangle \\ &= \langle 5, 1\rangle.\end{aligned}$$

7. (15 points) Solve the triangle or explain why it cannot be solved.



Solution: We cannot use law of sines (all side-angle pairs contain an unknown), so we must use law of cosines.

Find b

By law of cosines,

$$b^2 = 5.2^2 + 4.4^2 - 2(5.2)(4.4)\cos(88°).$$

Therefore

$$b = \sqrt{5.2^2 + 4.4^2 - 2(5.2)(4.4)\cos(88°)} \approx \sqrt{44.8} \approx 6.69.$$

Find γ

By law of cosines,

$$5.2^2 = 4.4^2 + 6.69^2 - 2(4.4)(6.69)\cos(\gamma).$$

Solving for $\cos(\gamma)$ yields

$$\cos(\gamma) = \frac{5.2^2 - 4.4^2 - 6.69^2}{-2(4.4)(6.69)} \approx 0.6297.$$

Therefore

$$\gamma = \cos^{-1}(0.6297) \approx 0.88 \text{ rad} = 50.97°.$$

Find α

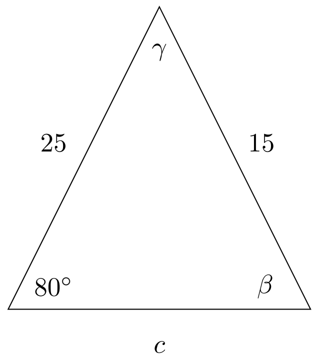
Using the fact that the sum of angles in a triangle is 180°, we see that

$$88° + 50.97° + \alpha = 180°,$$

i.e.

$$\alpha = 180° - 88° - 50.97° = 41.03°.$$

8. (15 points) Solve the triangle or explain why it cannot be solved.



Solution: Since we know a side-angle pair we can use law of sines. By law of sines,

$$\frac{\sin(80°)}{15} = \frac{\sin(\beta)}{25}.$$

Therefore

$$\sin(\beta) = \frac{25\sin(80°)}{15} \approx 1.641.$$

We cannot find β because to do so would require computing $\sin^{-1}(1.641)$, but this produces an error since 1.641 is not in the domain of $\sin^{-1}$. Therefore this problem does not define a triangle.