MATH 3315 - EXAM 4 - FALL 2017
SOLUTION

Friday 17 November 2017
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (13 points) Find the exact value of the series.

   (a) (6 points) $\displaystyle\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$

   Solution: This is a geometric series with $r = \dfrac{2}{3}$. So compute

$$\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3.$$

   (b) (7 points) $\displaystyle\sum_{k=2}^{\infty}\left(\frac{1}{k} - \frac{1}{k-1}\right)$

   Solution: Consider the infinite sum as a limit of a partial sum:

$$\begin{aligned}
\sum_{k=2}^{\infty}\frac{(-1)^k x^k}{k} &= \lim_{N\to\infty}\sum_{k=2}^{N}\frac{(-1)^k x^k}{k} \\
&= \lim_{N\to\infty}\left(\frac{\cancel{1}}{\cancel{2}} - \frac{1}{1}\right) + \left(\frac{\cancel{1}}{\cancel{3}} - \frac{\cancel{1}}{\cancel{2}}\right) + \left(\frac{\cancel{1}}{\cancel{4}} - \frac{\cancel{1}}{\cancel{3}}\right) + \dots + \left(\frac{1}{N} - \cancel{\frac{1}{N-1}}\right) \\
&= \lim_{N\to\infty} -1 + \frac{1}{N}
\end{aligned}$$

2. (21 points) Find the interval of convergence and radius of convergence of the following power series.

   (a) (7 points) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{k}$

   Solution: Using the ratio test, compute

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{(-1)^{k+1}x^{k+1}}{k+1}\frac{k}{(-1)^k x^k}\right| &= \lim_{k\to\infty}\left|x\frac{k}{k+1}\right| \\
&= |x|.
\end{aligned}$$

   The ratio test tells us that the series converges provided that $|x| < 1$, i.e. $-1 < x < 1$ and that the series diverges provided that $|x| > 1$, i.e. $x < -1$ and $x > 1$. We must check the endpoints $x = 1$ and $x = -1$ manually. At $x = 1$, the series becomes $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$, which converges by the alternating series test. At $x = -1$, the series becomes $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k(-1)^k}{k} = \sum_{k=1}^{\infty}\frac{1}{k}$, which diverges (it is the harmonic series). Therefore the interval of convergence is $(-1, 1]$ and the radius of convergence is 1.

   (b) (7 points) $\displaystyle\sum_{k=1}^{\infty} k!x^k$

   Solution: By the ratio test,

$$\lim_{k\to\infty}\left|\frac{(k+1)!x^{k+1}}{k!x^k}\right| = \lim_{k\to\infty}|(k+1)x| = \begin{cases} 0, & x = 0 \\ \infty, & x \neq 0. \end{cases}$$

   Therefore the interval of convergence of the series is the set $[0, 0] = \{0\}$ and the radius of convergence is 0.

   (c) (7 points) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{(k+1)!}$

   Solution: By the ratio test,

$$\lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+2)!}\frac{(k+1)!}{x^k}\right| = \lim_{k\to\infty}\left|\frac{x}{k+2}\right| = 0,$$

   and so the interval of convergence of the series is $(-\infty, \infty) = \mathbb{R}$ and the radius of convergence is ∞.

3. (10 points) Find a power series for $f(x) = \dfrac{1}{2-x}$ centered at $c = 3$.

Solution: Think about a simple geometric series centered at $c = 3$:

$$\sum_{k=0}^{\infty}(x-3)^k = \frac{1}{1-(x-3)} = \frac{1}{1-x+3}.$$

So we want to rewrite $\dfrac{1}{2-x}$ to have a denominator with "$-x+3$" in it. Therefore write

$$\frac{1}{2-x} = \frac{1}{2-x+3-3} = \frac{1}{-1-(x-3)} = \frac{-1}{1-(3-x)} = -\sum_{k=0}^{\infty}(3-x)^k.$$

4. (56 points) Converge or diverge? Why? Explain.

(a) (7 points) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k}{k^2-5}$

Solution: Converges by the alternating series test:

$$\lim_{k\to\infty}\frac{k}{k^2-5} \overset{L.H.}{=} \lim_{k\to\infty}\frac{1}{2k} = 0.$$

(b) (7 points) $\displaystyle\sum_{k=1}^{\infty}\frac{k^4-3k^2+1}{2k^4-k+17}$

Solution: Diverges by the test for divergence:

$$\lim_{k\to\infty}\frac{k^4-3k^2+1}{2k^4-k+17} = \frac{1}{2} \neq 0.$$

(c) (7 points) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^e}$

Solution: Converges because it is a p-series with $p = e \approx 2.71\ldots > 1$.

(d) (7 points) $\displaystyle\sum_{k=1}^{\infty}\frac{5^k}{(2k)!}$

Solution: Converges by the ratio test:

$$\lim_{k\to\infty}\left|\frac{5^{k+1}}{(2(k+1))!}\frac{(2k)!}{5^k}\right| = \lim_{k\to\infty}\frac{5}{(2k+2)(2k+1)} = 0 < 1.$$

(e) (7 points) $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{(3k+1)^k}$

Solution: Converges by the root test:

$$\lim_{k\to\infty}\sqrt[k]{\left|\frac{k^k}{(3k+1)^k}\right|} = \lim_{k\to\infty}\frac{k}{3k+1} = \frac{1}{3} < 1.$$

(f) (7 points) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{10}{3}\right)^k$

Solution: Diverges – this is a geometric series with $r = \dfrac{10}{3} > 1$.

(g) (7 points) $\sum_{k=1}^{\infty} \frac{k}{e^{k^2}}$

Solution: Converges by the integral test:

$$\int_1^{\infty} \frac{x}{e^{x^2}} \mathrm{d}x \overset{u=x^2}{=} \frac{1}{2} \int_1^{\infty} e^{-u} \mathrm{d}u = \frac{1}{2}[-0 + e^{-1}] < 1.$$

(h) (7 points) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2 - 1}}$

Solution: Diverges by the limit comparison test. Let $b_k = \frac{1}{k}$. Then compute

$$\lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2-1}}} = \lim_{k\to\infty} \frac{\sqrt{k^2-1}}{k} = \lim_{k\to\infty} \sqrt{1 - \frac{1}{k^2}} = \sqrt{1} = 1.$$

The result follows because $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges as it is the harmonic series.