First order arithmetic

Axioms

1. $(\forall x)(0 \neq Sx)$
2. $(\forall x)(\forall y)(Sx = Sy \rightarrow x = y)$
3. $(\forall y)(y = 0 \vee (\exists x)(Sx = y))$
4. $(\forall x)(x + 0 = x)$
5. $(\forall x)(\forall y)(x + Sy = S(x + y))$
6. $(\forall x)(x \cdot 0 = 0)$
7. $(\forall x)(\forall y)(x \cdot Sy = (x \cdot y) + x)$
8. $(\forall x)(\forall y)(x + y = y + x)$
9. $(\forall x)(\forall y)(x \cdot y = y \cdot x)$

First order arithmetic

Prove in 1st order arithmetic: $(x+S0)=SS0 \leftrightarrow x=S0$.

$\{\mathrm{Ax}\,5\}$	(1) $(\forall y)(x+Sy=S(x+y))$	Axiom 5 US
$\{\mathrm{Ax}\,5\}$	(2) $x+S0=S(x+0)$	1 US
$\{\mathrm{Ax}\,4\}$	(3) $x+0=x$	Axiom 4 US
$\{\mathrm{Ax}\,4,5\}$	(4) $x+S0=Sx$	2 3 Identity Law
$\{5\}$	(5) $(x+S0)=SS0$	Premise
$\{5,\mathrm{Ax}\,4,5\}$	(6) $SS0=Sx$	4 5 Identity Law
$\{\mathrm{Ax}\,2\}$	(7) $(\forall y)(SS0=Sy \rightarrow S0=y)$	Axiom 2 US
$\{\mathrm{Ax}\,2\}$	(8) $SS0=Sx \rightarrow S0=x$	7 US
$\{5,\mathrm{Ax}\,2,4,5\}$	(9) $S0=x$	6 8 Detachment
$\{5,\mathrm{Ax}\,2,4,5\}$	(10) $x=S0$	9 Identity
$\{\mathrm{Ax}\,2,4,5\}$	(11) $(x+S0)=SS0 \rightarrow x=S0$	5 10 C.P.
$\{12\}$	(12) $x=S0$	Premise
$\{12,\mathrm{Ax}\,5\}$	(13) $(x+S0)=S(S0+0)$	2 12 Identity Law
$\{\mathrm{Ax}\,4\}$	(14) $S0+0=S0$	Axiom 4
$\{12,\mathrm{Ax}\,4,5\}$	(15) $x+S0=SS0$	13 14 Identity Law
$\{\mathrm{Ax}\,4,5\}$	(16) $x=S0 \rightarrow (x+S0)=SS0$	12 15 C.P.
$\{\mathrm{Ax}\,2,4,5\}$	(17) $(x+S0)=SS0 \leftrightarrow x=S0$	11 16 Law of Bicondition

Sometimes it is useful to break a proof of a theorem down into constituent parts and then combine the parts. This can help "chunk" a longer proof into a sequence of simpler proofs. We could have proven $(x+S0)=SS0 \leftrightarrow x=S0$ in the following way:

- **Theorem A**: $(x+S0)=SS0 \rightarrow x=S0$
- **Theorem B**: $x=S0 \rightarrow (x+S0)=SS0$

Notice in the previous slide, **Theorem A** was proven on line (11) and depends on Axioms 2, 4, and 5 and **Theorem B** was proven on line (16) and depends on Axioms 4 and 5.

We allow ourselves to use theorems in a formal deduction by placing it on a new line (with proper documentation).

First order arithmetic

Prove

$$(x + S0) = SS0 \leftrightarrow x = S0$$

from **Theorem A** and **Theorem B**:

$\{\mathrm{Ax}\, 2, 4, 5\}$	(1) $(x + S0) = SS0 \rightarrow x = S0$	**Theorem A**
$\{\mathrm{Ax}\, 4, 5\}$	(2) $x = S0 \rightarrow (x + S0) = SS0$	**Theorem B**
$\{\mathrm{Ax}\, 2, 4, 5\}$	(3) $(x + S0) = SS0 \leftrightarrow x = S0$	1 2 Law of Biconditionals

Prove $(\forall x)(x \cdot (x + S0) = x \cdot x + x)$ in first order arithmetic.

$\{\text{Axiom } 7\}$	(1) $(\forall y)(x \cdot Sy = (x \cdot y) + x$	Axiom 7 US
$\{\text{Axiom } 7\}$	(2) $x \cdot Sx = (x \cdot x) + x$	1 US
$\{\text{Axiom } 5\}$	(3) $(\forall y)(x + Sy = S(x + y))$	Axiom 5 US
$\{\text{Axiom } 5\}$	(4) $x + S0 = S(x + 0)$	3 US
$\{\text{Axiom } 4\}$	(5) $x + 0 = x$	Axiom 4
$\{\text{Axiom } 4, 5\}$	(6) $x + S0 = Sx$	4 5 Identity Law
$\{\text{Axiom } 4, 5, 7\}$	(7) $x \cdot (x + S0) = (x \cdot x) + x$	2 6 Identity Law
$\{\text{Axiom } 4, 5, 7\}$	(8) $(\forall x)(x \cdot (x + S0) = (x \cdot x) + x)$	7 UG