Homework 10

1. Prove the following in pure identity theory or give an interpretation to show it is not true:

a.) $(x = y \wedge (y = z \wedge z = w)) \to x = w$

Solution:

$\{1\}$	(1) $(x = y \wedge (y = z \wedge z = w))$	Premise
$\{1\}$	(2) $x = y$	1 Simplification
$\{1\}$	(3) $y = z$	1 Commutative law of $\wedge$, Simplification twice
$\{1\}$	(4) $z = w$	1 Com. of $\wedge$, Simpl., Com. of $\wedge$, Simpl.
$\{1\}$	(5) $x = z$	2 3 Identity
$\{1\}$	(6) $x = w$	4 5 Identity
$\{\}$	(7) $(x = y \wedge (y = z \wedge z = w)) \to x = w$	1 6 C.P.

b.) $\neg(x = y) \wedge \neg(y = z) \to \neg(x = z)$

Solution: Let $\mathscr{U} = \{0, 1\}$. Then assign $x = 0$, $y = 1$, and $z = 0$. In this interpretation, $\neg(x = y)$ is true, $\neg(y = z)$ is true, but $\neg(x = z)$ is false (because, in fact $x = z$).

2. Prove the following in first order arithmetic or give an interpretation to show it is not true:

a.) $S0 \cdot S0 = S0$

Solution:

$\{\text{Axiom } 7\}$	(1) $S0 \cdot S0 = (S0 \cdot 0) + S0$	Axiom 7, then US twice, 1st setting $x = S0$ and 2nd setting $y = 0$
$\{\text{Axiom } 6\}$	(2) $S0 \cdot 0 = 0$	Axiom 6, then US with $x = S0$
$\{\text{Axioms } 6, 7\}$	(3) $S0 \cdot S0 = 0 + S0$	1 2 Identity
$\{\text{Axiom } 8\}$	(4) $S0 + 0 = 0 + S0$	Axiom 8, then US twice, 1st setting $x = S0$ and 2nd setting $y = 0$
$\{\text{Axiom } 4\}$	(5) $S0 + 0 = S0$	Axiom 4 then US with $x = S0$
$\{\text{Axiom } 4, 8\}$	(6) $S0 = 0 + S0$	4 5 Identity
$\{\text{Axiom } 4, 6, 7, 8\}$	(7) $S0 \cdot S0 = S0$	3 6 Identity

b.) $S0 \cdot SS0 = S0 + S0$

Solution: We will use the following theorem in this proof (this theorem was proven above):

Theorem A: $S0 \cdot S0 = S0$

$\{\text{Axioms } 4, 6, 7, 8\}$	(1) $S0 \cdot S0 = S0$	Theorem A
$\{\text{Axiom } 7\}$	(2) $S0 \cdot SS0 = (S0 \cdot S0) + S0$	Axiom 7, then US twice
$\{\text{Axioms } 4, 6, 7, 8\}$	(3) $S0 \cdot SS0 = S0 + S0$	1 2 Identity

c.) $SS0 \cdot SSS0 = SSSS0$

Solution: Consider the interpretation $\mathscr{U} = \{2, 3, 4\}$ with $SS0 = 2$, $SSS0 = 3$, $SSSS0 = 4$, "+" means "usual addition", and "$\cdot$" means "usual multiplication (this interpretation satisfies all the axioms of first order arithmetic – as discussed in class, that did not need to be explicitly mentioned in the homework). Then the sentence $SS0 \cdot SSS0 = SSSS0$ reads "$2 \cdot 3 = 4$" in our interpretation, which is **false**.

3. Consider a new theory called "total order theory" which has a two-place predicate "$\leq$" and the following axioms:

Axiom 1	$(\forall x)(\forall y)(\forall z)(x \leq y \wedge y \leq z \to x \leq z)$
Axiom 2	$(\forall x)(x \leq x)$
Axiom 3	$(\forall x)(\forall y)(x \leq y \wedge y \leq x \to x = y$
Axiom 4	$(\forall x)(\forall y)(x \leq y \vee y \leq x)$

Prove the following in total order theory or give an interpretation to show it is not true:

a.) $(\forall x)(\forall y)(\neg(x \leq y) \to y \leq x)$

Solution:

$\{1\}$	(1) $\neg(x \leq y)$	Premise
$\{\text{Axiom } 4\}$	(2) $x \leq y \vee y \leq x$	Axiom 4, then US twice
$\{1, \text{Axiom } 4\}$	(3) $y \leq x$	1 2 Modus tollendo ponens
$\{\text{Axiom } 4\}$	(4) $\neg(x \leq y) \to y \leq x$	1 3 C.P.
$\{\text{Axiom } 4\}$	(5) $(\forall x)(\forall y)(\neg(x \leq y) \to y \leq x)$	3 UG twice

b.) $x \leq y \wedge \neg(x \leq z) \to \neg(y \leq z)$

Solution: Define the following theorem (proved in part .a)):

Theorem A: $(\forall x)(\forall y)(\neg(x \leq y) \rightarrow y \leq x)$

{1}	(1) $x \leq y \wedge \neg(x \leq z)$	Premise
{1}	(2) $x \leq y$	1 Simplification
{1}	(3) $\neg(x \leq z)$	1 Commutative law of $\wedge$, Simplification
{Axiom 1}	(4) $x \leq y \wedge y \leq z \rightarrow x \leq z$	Axiom 1, US three times
{Axiom 1}	(5) $\neg(x \leq z) \rightarrow \neg(x \leq y \wedge y \leq z)$	4 Contraposition
{Axiom 1}	(6) $\neg(x \leq z) \rightarrow \neg(x \leq y) \vee \neg(y \leq z)$	5 ET (DeMorgan's law)
{1, Axiom 1}	(7) $\neg(x \leq y) \vee \neg(y \leq z)$	3 6 Detachment
{1}	(8) $\neg(\neg(x \leq y))$	2 Double negation
{1, Axiom 1}	(9) $\neg(y \leq z)$	7 8 Modus tollendo ponens
{Axiom 1}	(10) $x \leq y \wedge \neg(x \leq z) \rightarrow \neg(y \leq z)$	1 9 C.P.