Homework 6 - MATH 2200 Spring 2017

1. Write the sentence using quantifiers and predicates. You are welcome to specify a universe in each part to simplify your formulas. Clearly specify the predicates you use.

(a) "All real numbers are complex numbers."

(b) "The sum of the angles in any triangle is 180°." (of course this is only true in "Euclidean geometry" – in "hyperbolic geometry" the sum of angles is strictly less than 180°)

(c) "All Lipschitz functions with compact support are absolutely continuous and of bounded variation."

(d) "There is a continuous function that is uniformly-continuous but is not Hölder-continuous."

(e) "Every even integer greater than 2 can be expressed as the sum of two prime numbers." (this is the famous "Goldbach's conjecture"...unknown if it is true or not!)

2. Let R be the predicate "is a real number" and let I be the predicate "is an integer". Is the sentence true or false? If false, explain why.

(a) $(\forall x)(Rx \to x^2 \geq 0)$

(b) $(\exists x)(Ix \wedge x = \sqrt{2})$

(c) $(\forall x)(x^2 + y^2 = 1 \to x^2 = 1 - y^2)$

(d) $(\forall x)((x > 3) \to (\exists y)(Iy \wedge x + y = \frac{1}{2}))$

3. Consider the possible universes of quantification: the natural numbers $\mathbb{N}$ (i.e. the numbers $0, 1, 2, \ldots$), the integers $\mathbb{Z}$ (i.e. the numbers $\ldots -1, 0, 1, 2, \ldots$), the real numbers $\mathbb{R}$ (i.e. everything on the number line), and complex numbers $\mathbb{C}$ (i.e. the numbers $a + bi$ where $i = \sqrt{-1}$ and a and b are real numbers).
List all universes (if any) in which the given formula is true.

(a) Let Pxy denote "is a solution of $x + 2 = y$": $(\forall x)(\exists y)(Pxy)$

(b) Let Qxy denote "is a solution of $2x + 3y = 1$": $(\forall y)(\exists x)(Qxy)$

(c) Let Rx denote "is a solution of $x^2 + 2 = 0$": $(\exists x)(Rx)$

(d) Let Exy denote "x is equal to y" and $Bzxy$ denote "z is a number between x and y":
$(\forall x)(\forall y)(\neg Exy \to (\exists z)(Bzxy))$

4. Diagram the scope of every quantifier that appears in the symbolized form of the sentence. Explain whether each instance of a variable is bound or unbound.

(a) "for all x, there exists a y such that $y < z$ and $x < y$" can be symbolized (with appropriate predicate symbols) as:

$$(\forall x)(\exists y)(Lyz \wedge Lxy)$$

(b) The definition of compactness in topology states
"For every open cover u there is a finite subcover f of u,"
and can be symbolized (with appropriate predicate symbols) as:

$$(\forall u)(Ou \to (\exists f)(Sfu))$$

(c) Hölder's inequality in L^p spaces states
"For all measurable functions f and g, for all $p \geq 1$ and for all $q \geq 1$, if $\frac{1}{p} + \frac{1}{q} = 1$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$,"
and can be symbolized (with appropriate predicate symbols) as:

$$(\forall f)(\forall g)\,[(Mf \wedge Mg) \to [(\forall p)(\forall q)[(Gp \wedge Gq) \to (Spq \to Lfg)]]]$$