MATH 2200 - EXAM 3 SPRING 2017
SOLUTION

Thursday 27 April 2017
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (8 points) Translate the following arguments into symbols. Clearly identify the premises and the conclusion.
   (a) (4 points) "All politicans are corrupt people. Some politicians have good intentions. Therefore some corrupt people have good intentions."
   Solution: The premises are the first two sentences and may be symbolized as $(\forall x)(Px \to Cx)$ and $(\exists x)(Px \wedge Gx)$. The conclusion is the third sentence and may be symbolized as $(\exists x)(Cx \wedge Gx)$.
   (b) (4 points) "No irrational number is rational. All transcendental numbers are irrational. Therefore no transcendental number is rational."
   Solution: The premises are the first two sentences and may be symbolized as $(\forall x)(Ix \to \neg Rx)$ and $(\forall x)(Tx \to Ix)$. The conclusion is the third sentence and may be symbolized as $(\forall x)(Tx \to \neg Rx)$.

2. (17 points) Presburger arithmetic is a "fragment" of first order arithmetic with a one-place predicate S, a two-place predicate $+$, and the following axioms:

   Axiom P1 $(\forall x)\neg(0 = Sx)$

   Axiom P2 $(\forall x)(\forall y)(Sx = Sy \to x = y)$

   Axiom P3 $(\forall x)(x + 0 = x)$

   Axiom P4 $(\forall x)(\forall y)(x + Sy = S(x + y))$

   Axiom P5 (Induction Schema) For any one-place predicate Px, the following is an axiom:

$$(P(0) \wedge (\forall x)(Px \to P(Sx))) \to (\forall y)(Py)$$

   (a) (10 points) Prove the following formula is a theorem of Presburger arithmetic:

$$S0 + SS0 = SSS0.$$

   You may use the following theorem (which depends on Axioms $P3$ and $P4$):

   Theorem A: $S0 + S0 = SS0$

   Solution:

$\{\text{Ax.P3, P4}\}$	(1) $S0 + S0 = SS0$	**Theorem A**
$\{\text{Ax.P4}\}$	(2) $S0 + SS0 = S(S0 + S0)$	Axiom P4
$\{\text{Ax.P3, P4}\}$	(3) $S0 + SS0 = SSS0$	1 2 Identity

   (b) (7 points) Write an interpretation to show that the following formula is not a theorem of Presburger arithmetic (if you use a "usual" arithmetical interpretation, you do not have to check the axioms):

$$S0 + S0 = SSS0.$$

   Solution: Define the universe $\mathscr{U} = \{1, 3\}$, assign the predicate "”+” the meaning of usual addition, let $S0 = 1$, and let $SSS0 = 3$. Then the sentence in question is interpreted as "$1 + 1 = 3$", which is false.

3. (17 points) Pure identity theory is the theory with an empty set of axioms (i.e. no axioms).
   (a) (10 points) Prove the following theorem of pure identity theory:

$$\neg(x = y) \wedge (y = z) \to \neg(x = z)$$

$\{1\}$	(1) $\neg(x = y) \wedge (y = z)$	Premise
$\{1\}$	(2) $\neg(x = y)$	1 Simplification
$\{1\}$	(3) $y = z$	1 Commutative Law of $\wedge$ and Simplification
$\{1\}$	(4) $\neg(x = z)$	2 3 Identity
$\{\}$	(5) $\neg(x = y) \wedge (y = z) \to \neg(x = z)$	1 4 C.P.

(b) (7 points) Write an interpretation to show that the following formula is not a theorem of pure identity theory:

$$\neg(x = y) \wedge (y = z) \to (x = z)$$

Solution: Define the universe to be $\mathscr{U} = \{1, 2\}$. Let $x = 1$, $y = 2$, and $z = 2$. Then $\neg(x = y)$ and $y = z$ are both true, while $x = z$ is false.

4. (17 points) Differential algebra theory has a one-place predicate ∂, two two-place predicates $+$ and $\cdot$, and obeys the following two axioms:

Axiom D1 $(\forall x)(\forall y)(\partial(x \cdot y) = (\partial x) \cdot y + x \cdot (\partial y))$

Axiom D2 $(\forall x)(\forall y)(\partial(x + y) = \partial(x) + \partial(y))$

Prove the following formula is a theorem of differential algebra theory:

$$\partial(f \cdot g + h \cdot k) = ((\partial f) \cdot g + f \cdot (\partial g)) + ((\partial h) \cdot k + h \cdot (\partial k)).$$

Solution:

$\{\text{Ax.D1}\}$	(1) $\partial(f \cdot g) = (\partial f) \cdot g + f \cdot (\partial g)$	Axiom D1, US twice
$\{\text{Ax.D1}\}$	(2) $\partial(h \cdot k) = (\partial h) \cdot k + h \cdot (\partial k)$	Axiom D1, US twice
$\{\text{Ax.D2}\}$	(3) $\partial(f \cdot g + h \cdot k) = \partial(f \cdot g) + \partial(h \cdot k)$	Axiom D2 US twice
$\{\text{Ax.D1}, \text{D2}\}$	(4) $\partial(f \cdot g + h \cdot k) = ((\partial f) \cdot g + f \cdot (\partial g)) + ((\partial h) \cdot k + h \cdot (\partial k))$	1, 2, 3 Identity

5. (8 points) Circle all errors in the following deduction:

$\{1\}$	(1)	$(\forall x)(Ax \to \neg Bx)$	Premise
$\{2\}$	(2)	$(\exists x)(Ax \wedge Cx)$	Premise
$\{1\}$	(3)	$Ax \to \neg Bx$	1 US
$\{2\}$	(4)	$\boxed{Ax \wedge Cx}$	2 ES
$\{2\}$	(5)	Ax	4 Simplification
$\{2\}$	⑤	Cx	4 Commutative law of $\wedge$ and Simplification
$\{1, 2\}$	(7)	$\neg Bx$	3 ④ Detachment
$\{1, 2\}$	(8)	$Cx \wedge \neg Bx$	6 7 Adjunction
$\{1, 2, ⑧\}$	(9)	$(\forall x)(Cx \wedge \neg Bx)$	8 UG

6. (17 points) Monoid theory is has a two-place predicate $*$ and is defined by the following axioms:
Axiom M1: $(\forall x)(\forall y)(\forall z)(x * (y * z) = (x * y) * z)$
Axiom M2: $(\exists e)(\forall x)(e * x = x \wedge x * e = x)$
Prove the following theorem of monoid theory:

$$(\exists e)(\forall x)(\forall y)(x * (y * e) = x * y)$$

Solution:

$\{\text{Ax.M2}\}$	(1) $\alpha * (x * y) = (x * y) \wedge (x * y) * \alpha = (x * y)$	Axiom M2, ES once, US once
$\{\text{Ax.M2}\}$	(2) $(x * y) * \alpha = (x * y)$	1 Commutative Law of $\wedge$ and Simplification
$\{\text{Ax.M1}\}$	(3) $x * (y * \alpha) = (x * y) * \alpha$	Axiom M1, US 3 times
$\{\text{Ax.M1}, \text{M2}\}$	(4) $x * (y * \alpha) = (x * y)$	1 2 Identity

7. (16 points) Consider the sets $w = \{(1, 5), (2, 5)\}$ and $q = \{(1, 5), (1, 4), (17, 17)\}$.

(a) (2 points) Compute $w \cup q$.
Solution: $w \cup q = \{(1, 5), (2, 5), (1, 4), (17, 17)\}$

(b) (2 points) Compute and $w \cap q$.
Solution: $w \cap q = \{(1, 5)\}$

(c) (2 points) Is w a function?
Solution: Yes.

(d) (2 points) Is q a function?
Solution: No – both $(1, 5)$ and $(1, 4)$ have the same first coordinate but different second coordinates.

(e) (2 points) Is $w \cup q$ a function?
Solution: No, for the same reason as (d).

(f) (2 points) Is $w \cap q$ a function?
Solution: Yes.

(g) (2 points) Is the set $\{(\emptyset, \emptyset), (\{\emptyset\}, \emptyset)\}$ a function?
Solution: Yes – no first coordinate is repeated.

(h) (2 points) Is $\emptyset$ a function?
Solution: Yes – this is "vacuously true", meaning it is true "for all elements of $\emptyset$" (because there are none).

TABLE OF USEFUL TAUTOLOGIES

TAUTOLOGICAL IMPLICATIONS

Law of Detachment	$P \,\&\, (P \to Q) \to Q$
Modus tollendo tollens	$-Q \,\&\, (P \to Q) \to -P$
Modus tollendo ponens	$-P \,\&\, (P \lor Q) \to Q$
Law of Simplification	$P \,\&\, Q \to P$
Law of Adjunction	$P \,\&\, Q \to P \,\&\, Q$
Law of Hypothetical Syllogism	$(P \to Q) \,\&\, (Q \to R) \to (P \to R)$
Law of Exportation	$[P \,\&\, Q \to R] \to [P \to (Q \to R)]$
Law of Importation	$[P \to (Q \to R)] \to [P \,\&\, Q \to R]$
Law of Absurdity	$[P \to Q \,\&\, -Q] \to -P$
Law of Addition	$P \to P \lor Q$

TAUTOLOGICAL EQUIVALENCES

Law of Double Negation	$P \leftrightarrow --P$
Law of Contraposition	$(P \to Q) \leftrightarrow (-Q \to -P)$
De Morgan's Laws	$-(P \,\&\, Q) \leftrightarrow -P \lor -Q$
	$-(P \lor Q) \leftrightarrow -P \,\&\, -Q$
Commutative Laws	$P \,\&\, Q \leftrightarrow Q \,\&\, P$
	$P \lor Q \leftrightarrow Q \lor P$
Law of Equivalence for Implication and Disjunction	$(P \to Q) \leftrightarrow -P \lor Q$
Law of Negation for Implication	$-(P \to Q) \leftrightarrow P \,\&\, -Q$
A Law for Biconditional Sentences	$(P \leftrightarrow Q) \leftrightarrow (P \to Q) \,\&\, (Q \to P)$
Another Law for Biconditional Sentences	$(P \leftrightarrow Q) \leftrightarrow (P \,\&\, Q) \lor (-P \,\&\, -Q)$

TWO FURTHER TAUTOLOGIES

Law of Excluded Middle	$P \lor -P$
Law of Contradiction	$-(P \,\&\, -P)$