# MATH 2200 - EXAM 1 SPRING 2017
# SOLUTION

Thursday 9 February 2017
Instructor: Tom Cuchta

**Instructions:**

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

Tautological implications and equivalences
**Commutative Law of** $\vee$: $P \vee Q \leftrightarrow Q \vee P$
**Law of Equivalence for Implication and Disjunction**: $(\neg P \vee Q) \leftrightarrow (P \to Q)$
**Law of Contraposition**: $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$
**Modus tollendo ponens**: $[(\neg P) \wedge (P \vee Q)] \to Q$
**Modus tollendo tollens**: $[(\neg Q) \wedge (P \to Q)] \to \neg P$
**Law of Detachment**: $[P \wedge (P \to Q)] \to Q$

1. (9 points) Translate the following sentences into symbolic notation, with letters standing only for atomic sentences. Be sure to identify which sentences correspond to which letters you use.
**This problem resembles #3(h) in HW1.**
   (a) (3 points) "The sum is positive or the sum is zero."
   *Solution:* Let $P$ be the sentence "The sum is positive." and let $Q$ be the sentence "The sum is zero". The given sentence, written symbolically, is $P \vee Q$.
   (b) (3 points) "If $f$ is midpoint-convex and $f$ is continuous, then $f$ is convex."
   *Solution:* Let $P$ be the sentence "$f$ is midpoint-convex." and let $Q$ be the sentence "$f$ is convex". Then the given sentence, written symbolically, is $P \to Q$.
   (c) (3 points) "The graph $G$ is a tree if and only if $G$ is connected and $G$ does not contain a cycle."
   *Solution:* Let $P$ be the sentence "The graph $G$ is a tree.", let $Q$ be the sentence "$G$ does not contain a cycle.", and let $R$ be the sentence "$G$ contains a cycle". The given sentence, written symbolically, is $P \leftrightarrow (Q \wedge \neg R)$.
   (*note: if you chose $R$ to instead be "$G$ does not contain a cycle", then you would have gotten* $P \leftrightarrow (Q \wedge R)$, which is ok)

2. (16 points) If $P$ and $Q$ are true and $R$ is false, then are the following sentences true or false?
**This problem resembles #4(a), #5(d), #6(e), and #6(i) in HW1.**
   (a) (4 points) $P \vee R$
   *Solution:* True (because $P$ is true)
   (b) (4 points) $\neg(\neg R)$
   *Solution:* False (because $\neg R$ is true, which is the case because $R$ is false)
   (c) (4 points) $R \to \neg P$
   *Solution:* True (because "False $\to$ anything" is always true)
   (d) (4 points) $R \leftrightarrow [P \to (Q \wedge R)]$
   *Solution:* True ($Q \wedge R$ is false, meaning $P \to (Q \wedge R)$ is false since $P$ is true. This means both $R$ and $P \to (Q \wedge R)$ are both false, making the biconditional true, since they match.)

3. (12 points) Use a truth table to determine whether the given sentence is a tautology or not.
**This problem resembles #10(c) and #10(i) in HW2.**
   (a) (6 points) $[P \wedge (\neg Q)] \vee Q$
   *Solution:*

| $P$ | $Q$ | $\neg Q$ | $P \wedge (\neg Q)$ | $[P \wedge (\neg Q)] \vee Q$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ |

   From the final column, we see that this is **not** a tautology.
   (b) (6 points) $[\neg P \wedge (P \vee Q)] \to Q$
   *Solution:*

| $P$ | $Q$ | $\neg P$ | $P \vee Q$ | $[\neg P \wedge (P \vee Q)]$ | $[\neg P \wedge (P \vee Q)] \to Q$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |

   From the final column, we that this **is** a tautology.

4. (12 points) **This problem resembles #16(c) in HW2.**

   (a) (6 points) Does the sentence $P \wedge (P \to Q)$ tautologically imply the sentence $Q$? If so, prove it using a truth table. If not, explain why not.
   *Solution:* Yes it does. To see it, we must show that $[P \wedge (P \to Q)] \to Q$ is a tautology.

| $P$ | $Q$ | $P \to Q$ | $P \wedge (P \to Q)$ | $[(P \to Q)] \to Q$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ |

   (b) (6 points) Does the sentence $\neg P \wedge (P \to Q)$ tautologically imply the sentence $P$? If so, prove it using a truth table. If not, explain why not.
   *Solution:* It does not. We show it via truth table:

| $P$ | $Q$ | $\neg P$ | $P \to Q$ | $\neg P \wedge (P \to Q)$ | $[\neg P \wedge (P \to Q)] \to P$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |

   Since the last table shows that the implication $[\neg P \wedge (P \to Q)] \to P$ is **not** a tautology, we say that $\neg P \wedge (P \to Q)$ does **not** tautologically imply $P$.

5. (14 points) **This problem resembles #17(b) and #17(d) in HW2.**

   (a) (7 points) Is the sentence $Q \vee \neg P$ tautologically equivalent to $Q \vee P$? If so, prove it using a truth table. If not, explain why not.
   *Solution:* It is not. We demonstrate via truth table:

| $P$ | $Q$ | $\neg P$ | $Q \vee \neg P$ | $Q \vee P$ | $(Q \vee \neg P) \to (Q \vee P)$ | $(Q \vee P) \to (Q \vee \neg P)$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $T$ |

   Since the last two columns do not exactly match, we see that they are not tautologically equivalent. *note: we could have also made another column for $(Q \vee P) \leftrightarrow (Q \vee \neg P)$ which would show that that sentence is not a tautology — this also shows the two sentences are not tautologically equivalent.*

   (b) (7 points) Is the sentence $P$ tautologically equivalent to the sentence $\neg(\neg P)$? If so, prove it using a truth table. If not, explain why not.

6. (9 points) Make a counterexample to show the following rule of inference is invalid: "from $\neg Q$ and $P \to \neg Q$, we may conclude $P$."
   **This problem resembles #1 in HW3.**
   *Solution:* We must pick truth values for $P$ and $Q$ that make the premises true and the conclusion false. So we let $Q$ be false (so that the premise $\neg Q$ is true) and we will let $P$ be false (so that the conclusion will be false). Note that doing this makes the other premise $P \to \neg Q$ to also be false.

   This means we have found a counterexample because our premises $\neg Q$ and $P \to \neg Q$ are true while simultaneously the conclusion $P$ is false (based on how we chose the truth values for $P$ and $Q$).

7. (12 points) There **three** errors in the following formal deduction. Find them and explain why they are errors:

**This problem resembles #2 in HW3.**

| | | |
|---|---|---|
| $\{1\}$ | (1) $P$ | Premise |
| $\{2\}$ | (2) $P \to \neg Q$ | Premise |
| $\{3\}$ | (3) $R \to Q$ | Premise |
| $\{3\}$ | (4) $\neg Q \to \neg R$ | 3 T (Law of Contraposition) |
| $\{1, 2\}$ | (5) $\neg Q$ | 1,2 T (Modus tollendo ponens) |
| $\{1, 2, 3, 4, 5\}$ | (7) $\neg R$ | 4, 5 T (Law of Detachment) |

*Solution:*

**First error**: Line (5) claims that it follows from lines 1 and 2 using modus tollendo ponens. This is not true, it actually follows from the Law of Detachment.

**Second error:** The left columns of the last line of the deduction claims that this line depends on the premises $\{1, 2, 3, 4, 5\}$, but lines 4 and 5 are not premises, and so they should not be listed here.

**Third error:** The last line of the deduction is called (7), but it should be (6).

8. (16 points) Construct a formal deduction for the following arguments. Do so by applying the stated rule(s).

**This problem resembles #4 and #5 in HW3.**

(a) (8 points) "If $n$ is odd, then $i^n$ is complex. In actuality, $i^n$ is not complex. Therefore $n$ is not odd."
**Rules:** Use **Modus tollendo tollens**.
*Solution:* Let $P$ be the sentence "$n$ is odd", let $Q$ be the sentence "$i^n$ is complex". We may write a formal deduction for this argument as follows:

| | | |
|---|---|---|
| $\{1\}$ | (1) $P \to Q$ | Premise |
| $\{2\}$ | (2) $\neg Q$ | Premise |
| $\{1, 2\}$ | (3) $\neg P$ | 1, 2 T Modus tollendo tollens |

(b) (8 points) "If the axiom of choice holds, then nonmeasurable sets exist. If nonmeasurable sets exist, then the Banach-Tarski paradox is true. I will quit mathematics or the Banach-Tarski paradox is not true. The axiom of choice holds. Therefore I will quit mathematics."
**Rules:** Use the **Law of Detachment** to conclude "the Banach-Tarski paradox is true" and then use **Commutative Law of** $\vee$ to conclude "the Banach-Tarski paradox is not true or I will quit mathematics." From here, use one more law (which?) to conclude "If the Banach-Tarski paradox is true, then I will quit mathematics." After that, complete the deduction using the **Law of Detachment**.
*Solution:* Let

$$P = \text{“The axiom of choice holds.”}$$
$$Q = \text{“Nonmeasurable sets exist.”}$$
$$R = \text{“The Banach-Tarski paradox is true.”}$$
$$S = \text{“I will quit mathematics.”}$$

Construct the formal deduction as follows:

| | | |
|---|---|---|
| $\{1\}$ | (1) $P \to Q$ | Premise |
| $\{2\}$ | (2) $Q \to R$ | Premise |
| $\{3\}$ | (3) $S \vee (\neg R)$ | Premise |
| $\{4\}$ | (4) $P$ | Premise |
| $\{1, 4\}$ | (5) $Q$ | 1,4 T (Law of Detachment) |
| $\{1, 2, 4\}$ | (6) $R$ | 2,5 T (Law of Detachment) |
| $\{1, 2, 3, 4\}$ | (7) $(\neg R) \vee S$ | 3 T (Commutative Law of $\vee$) |
| $\{1, 2, 3, 4\}$ | (8) $R \to S$ | 7 T (Law of Equivalence for Implication and Disjunction) |
| $\{1, 2, 3, 4\}$ | (9) $S$ | 6, 8 T (Law of Detachment) |