MATH 1540 - EXAM 3 - FALL 2017

Name: ______________________________

2 November 2017
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (24 points) Prove the identity.

(a) (8 points) $\tan(x)\cos(x) = \sin(x)$

Solution: Recall that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$. Start on the left-hand side and compute

$$\begin{aligned}\tan(x)\cos(x) &= \frac{\sin(x)}{\cancel{\cos(x)}}\cancel{\cos(x)} \\ &= \sin(x),\end{aligned}$$

as was to be proved.

(b) (8 points) $\sin(x) - \sin^3(x) = \sin(x)\cos^2(x)$

Solution: Recall the Pythagorean identity $\cos^2(x) + \sin^2(x) = 1$ and rearrange it so that $\cos^2(x) = 1 - \sin^2(x)$. Start on the left-hand side and compute

$$\begin{aligned}\sin(x) - \sin^3(x) &= \sin(x)\left[1 - \sin^2(x)\right] \\ &= \sin(x)\cos^2(x),\end{aligned}$$

as was to be proved.

(c) (8 points) $\dfrac{\sin(x+h) - \sin(x)}{h} = \sin(x)\left(\dfrac{\cos(h) - 1)}{h}\right) + \cos(x)\left(\dfrac{\sin(h)}{h}\right)$

Solution: Recall the sum identity for sine: $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$. Now compute

$$\begin{aligned}\frac{\sin(x+h) - \sin(x)}{h} &= \frac{(\sin(x)\cos(h) + \cos(x)\sin(h)) - \sin(x)}{h} \\ &= \frac{\sin(x)[\cos(h) - 1] + \cos(x)\sin(h)}{h} \\ &= \sin(x)\frac{\cos(h) - 1}{h} + \cos(x)\frac{\sin(h)}{h},\end{aligned}$$

as was to be proved.

2. (14 points) Find the exact value.

(a) (7 points) $\sin\left(\dfrac{5\pi}{12}\right)$

Solution: Note that $\dfrac{5\pi}{12} = \dfrac{\pi}{4} + \dfrac{\pi}{6}$. Therefore use the sum identity for sine to compute

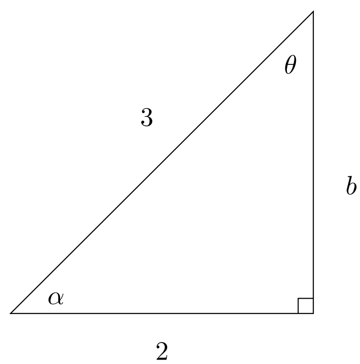
$$\begin{aligned}\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) &= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}.\end{aligned}$$

(b) (7 points) $\sin\left(\dfrac{7\pi}{8}\right)$

Solution: Recall the half-angle identity for sine: $\sin\left(\dfrac{\alpha}{2}\right) = \pm\sqrt{\dfrac{1 - \cos(\alpha)}{2}}$. Therefore since $\dfrac{7\pi}{8}$ is in quadrant II and if $\alpha = \frac{7\pi}{4}$ then $\frac{\alpha}{2} = \frac{7\pi}{8}$, compute

$$\begin{aligned}\sin\left(\frac{7\pi}{8}\right) &= \sin\left(\frac{\frac{7\pi}{4}}{2}\right)\\ &= +\sqrt{\frac{1-\cos(\frac{7\pi}{4})}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}.\end{aligned}$$

3. (14 points) Consider the following triangle:



Now find

(a) (7 points) $\sin\left(\frac{\alpha}{2}\right)$

Solution: First find b: using the Pythagorean theorem, we write $2^2 + b^2 = 3^2$ and so $b = \sqrt{9-4} = \sqrt{5}$. Therefore, using the half-angle identity for sine:

$$\sin\left(\frac{\alpha}{2}\right) = +\sqrt{\frac{1-\cos(\alpha)}{2}} = \sqrt{\frac{1-\frac{2}{3}}{2}}$$

(b) (7 points) $\tan(2\alpha)$

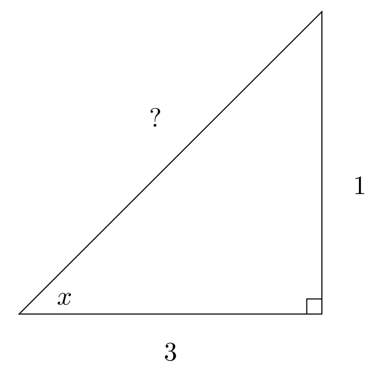
Solution: Recall the double angle identities for sine and cosine: $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ and $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ (using the other double angle identities for cosine are also OK). Now compute

$$\begin{aligned}\tan(2\alpha) &= \frac{\sin(2\alpha)}{\cos(2\alpha)}\\ &= \frac{2\sin(\alpha)\cos(\alpha)}{\cos^2(\alpha)-\sin^2(\alpha)}\\ &= \frac{2(\frac{\sqrt{5}}{3})(\frac{2}{3})}{\left(\frac{2}{3}\right)^2 - \left(\frac{\sqrt{5}}{3}\right)^2}\\ &= \frac{\frac{4\sqrt{5}}{9}}{\frac{4}{9}-\frac{5}{9}}\\ &= \frac{\frac{4\sqrt{5}}{9}}{\frac{1}{9}}\\ &= 4\sqrt{5}\end{aligned}$$

4. (16 points) If $\tan(x) = -\frac{1}{3}$ and x is in quadrant IV, find

(a) (8 points) $\sin(2x)$

Solution: We need to find $\sin(x)$ and $\cos(x)$. To do so, draw a triangle that agrees with $\tan(x) = \frac{1}{3}$:



To find the hypotenuse (labeled ?) we use the Pythagorean theorem: $3^2 + 1^2 = ?^2$ so that $? = \sqrt{10}$. Therefore $\cos(x) = \frac{3}{\sqrt{10}}$ and $\sin(x) = -\frac{1}{\sqrt{10}}$ (note: the negative is because the angle x is in quadrant IV!). Therefore using the double angle identity for sine,

$$\sin(2x) = 2\sin(x)\cos(x) = 2\left(-\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right).$$

(b) (8 points) $\cos\left(\frac{x}{2}\right)$

Solution: Using the half-angle identity for cosine,

$$\cos\left(\frac{x}{2}\right) = -\sqrt{\frac{1-\cos(x)}{2}} = -\sqrt{\frac{1-\frac{3}{\sqrt{10}}}{2}}.$$

5. (32 points) Solve the equation on $[0, 2\pi)$.

(a) (8 points) $2\cos(\theta) - 1 = 0$

Solution: Solve for $\cos(\theta)$ to get $\cos(\theta) = \frac{1}{2}$. From the unit circle, we observe that $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$.

(b) (8 points) $\sin^2(\theta) = \frac{3}{4}$

Solution: Take the square root of both sides of the equation (recall: when doing this, you **must** introduce a "±"):

$$\sin(\theta) = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}.$$

From the unit circle we see

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

(c) (8 points) $\cos(x) - \cos(x)\tan(x) = 0$

Solution: Factor $\cos(x)$ out of the left-hand side to get

$$\cos(x)(1 - \tan(x)) = 0.$$

Using the property of real numbers that $a \cdot b = 0$ implies $a = 0$ or $b = 0$, we break the trig equation down into $\cos(x) = 0$ and $1 - \tan(x) = 0$ (i.e. $\tan(x) = 1$). Therefore by the unit circle, $\cos(x) = 0$ has solution $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $\tan(x) = 1$ has solution $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

(d) (8 points) $\sin(2x) = \dfrac{1}{2}$

Solution: Let $\psi = 2x$. Then solving $\sin(\psi) = \dfrac{1}{2}$ yields the general solutions

$$\begin{cases} 2x &= \psi &= \dfrac{\pi}{6} + 2n\pi = \dfrac{\pi}{6} + \dfrac{12n\pi}{6} \\ 2x &= \psi &= \dfrac{5\pi}{6} + 2n\pi = \dfrac{5\pi}{6} + \dfrac{12n\pi}{6} \end{cases}$$

Dividing by 2 yields

$$\begin{cases} x &= \dfrac{\pi}{12} + \dfrac{12n\pi}{12} \\ x &= \dfrac{5\pi}{12} + \dfrac{12n\pi}{12} \end{cases}$$

Now we seek the values of $n = \ldots, -2, -1, 0, 1, 2, \ldots$ for which $0 \leq x < 2\pi$, or in other words $0 \leq x < \dfrac{24\pi}{12}$.

$$n = -1 : \begin{cases} x = \dfrac{\pi}{12} - \dfrac{12\pi}{12} = -\dfrac{11\pi}{12} & \times \\ x = \dfrac{5\pi}{12} - \dfrac{12\pi}{12} = -\dfrac{7\pi}{12} & \times \end{cases}$$

$$n = 0 : \begin{cases} x = \dfrac{\pi}{12} & \checkmark \\ x = \dfrac{5\pi}{12} & \checkmark \end{cases}$$

$$n = 1 : \begin{cases} x = \dfrac{\pi}{12} + \dfrac{12\pi}{12} = \dfrac{13\pi}{12} & \checkmark \\ x = \dfrac{5\pi}{12} + \dfrac{12\pi}{12} = \dfrac{17\pi}{12} & \checkmark \end{cases}$$

$$n = 2 : \begin{cases} x = \dfrac{\pi}{12} + \dfrac{24\pi}{12} = \dfrac{25\pi}{12} & \times \\ x = \dfrac{5\pi}{12} + \dfrac{24\pi}{12} = \dfrac{29\pi}{12} & \times \end{cases}$$

Therefore the solution of the original equation is $x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}$.