MATH 1115 - EXAM 1 - FALL 2016
SOLUTION

Thursday 15 September 2016
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (10 points) Circle **T** for true or **F** for false.

   (a) (2 points) (**T**) **F** 1 radian is a larger angle than 1 degree.
   Solution: To see this, convert one of the angles to another and compare:

$$1\,\text{radian} = (1\,\text{radian})\frac{180^\circ}{\pi\,\text{radians}} = \left(\frac{180}{\pi}\right)^\circ,$$

   and since π is approximately 3 (it is $3.1415\ldots$), this means $1\,\text{radian}$ is approximately 60° (actually it is 57.3°).

   (b) (2 points) (**T**) **F** An angle is formed when two rays share a common vertex.
   Solution: This is the definition on pg. 96 in the book.

   (c) (2 points) **T** (**F**) The fundamental period of a function f is the largest period of f.
   Solution: The definition on pg. 128 says that the fundamental period is the *smallest* period of f.

   (d) (2 points) **T** (**F**) The area of the sector of a circle of radius r subtended by θ degrees is given by $A = \frac{1}{2}r^2\theta$.
   Solution: The theorem on pg. 103 requires that θ be measured in radians. A similar situation occurs when discussing the arclength formula $s = r\theta$ – the θ must be measured in radians.

   (e) (2 points) (**T**) **F** The domain of sine is all real numbers.
   Solution: This is true; see the table on pg. 127.

2. (15 points) Do the following conversion problems.

   (a) (5 points) Convert 125° into radians.
   Solution: Compute

$$125^\circ = 125^\circ\frac{\pi\,\text{radians}}{180^\circ} = \frac{125\pi}{180}\,\text{radians} = \frac{25\pi}{36}\,\text{radians}.$$

   (b) (5 points) Convert $\frac{\pi}{7}$ radians into degrees.
   Solution: Compute

$$\frac{\pi}{7}\,\text{radians} = \left(\frac{\pi}{7}\,\text{radians}\right)\left(\frac{180^\circ}{\pi\,\text{radians}}\right) = \left(\frac{180}{7}\right)^\circ.$$

   (c) (5 points) Convert $\frac{1}{11}$ radians into degrees.
   Solution: Compute

$$\frac{1}{11}\,\text{radians} = \left(\frac{1}{11}\,\text{radians}\right)\left(\frac{180^\circ}{\pi\,\text{radians}}\right) = \left(\frac{180}{11\pi}\right)^\circ.$$

3. (15 points) Do the following problems.

   (a) (5 points) Find the radius of of a circle where an angle $\theta = \frac{1}{5}$ radians subtends an arc length of $s = 3$.
   Solution: Using the formula $s = r\theta$ (recall that θ **must** be measured in radians! See the Theorem on page 100 and discussion following it.), we get substitute the known value in and get

$$3 = r\frac{1}{5},$$

   and solve for r to get

$$15 = r.$$

   (b) (5 points) Find the angle that subtends an arc length of $s = 23$ of a circle with radius $r = 9$.
   Solution: Using $s = r\theta$ plug in the known information to get

$$23 = 9\theta,$$

   and solve for θ to get

$$\theta = \frac{23}{9} \text{ radians}.$$

   (c) (5 points) Find the arc length subtended by an angle of $\theta = 123°$ of a circle with radius $r = 5$.
   Solution: We **must** convert the angle measured in degrees to an angle measured in radians:

$$\theta = 123° = 123° \frac{\pi \text{ radians}}{180°} = \frac{123\pi}{180} \text{ radians} = \frac{41\pi}{60} \text{ radians}$$

4. (10 points) The point $\left(\frac{1}{5}, -\frac{2\sqrt{6}}{5}\right)$ is on the unit circle and corresponds to the angle θ. Find the value of the six trigonometric functions of θ.
Solution: From the definitions of sine and cosine (pg. 110) we immediately see that

$$\sin(\theta) = -\frac{2\sqrt{6}}{5},$$

and

$$\cos(\theta) = \frac{1}{5}.$$

From this we may deduce

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\frac{2\sqrt{6}}{5}}{\frac{1}{5}} = \left(-\frac{2\sqrt{6}}{5}\right)\left(\frac{5}{1}\right) = -2\sqrt{6},$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = -\frac{1}{2\sqrt{6}},$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{-\frac{2\sqrt{6}}{5}} = -\frac{5}{2\sqrt{6}},$$

and

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{\frac{1}{5}} = 5.$$

5. (10 points) The value of $\sin(\theta)$ and $\cos(\theta)$ are given. Find the value of the four remaining trigonometric functions: $\sin(\theta) = -\frac{3}{4}$ and $\cos(\theta) = \frac{\sqrt{7}}{4}$.
Solution: We proceed similar to number 4:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\frac{3}{4}}{\frac{\sqrt{7}}{4}} = -\frac{3}{\sqrt{7}},$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = -\frac{\sqrt{7}}{3},$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = -\frac{4}{3},$$

and

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{4}{\sqrt{7}}.$$

6. (10 points) Find the value of the five remaining trigonometric functions given that $\cos(\theta) = -\frac{3}{5}$ and θ is in quadrant II.
Solution: Using the Pythagorean identity (equation (5) on pg. 131) $\sin^2(\theta) + \cos^2(\theta) = 1$, we plug in $\cos(\theta) = -\frac{3}{5}$ to get

$$\sin^2(\theta) + \left(-\frac{3}{5}\right)^2 = 1,$$

and simplify the middle term to get

$$\sin^2(\theta) + \frac{9}{25} = 1.$$

Now subtract $\frac{9}{25}$ from both sides to get

$$\sin^2(\theta) = 1 - \frac{9}{25} = \frac{25}{25} - \frac{9}{25} = \frac{16}{25}.$$

Take the square root of both sides of the equation (don't forget that taking square roots of equations always yields a $\pm$) to get

$$\sin(\theta) = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}.$$

We must decide whether to take the value of $\sin(\theta)$ to be $\frac{4}{5}$ or $-\frac{4}{5}$. To make this decision, we recall that the problem told us that θ is in quadrant II – the $\sin(\theta)$ is **positive** whenever θ lies in quadrant II – and so we see that

$$\sin(\theta) = \frac{4}{5}.$$

From here we proceed like we did in problems 4 and 5:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{4}{3},$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} = -\frac{3}{4},$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{5}{4},$$

and

$$\sec(\theta) = \frac{1}{\cos(\theta)} = -\frac{5}{3}.$$

7. (10 points) Find the value of the five remaining trigonometric functions given that $\cot(\theta) = -\frac{3}{7}$ and $\cos(\theta) > 0$.

Solution: Using the Pythagorean identity (equation (7) on pg. 131) $\cot^2(\theta) + 1 = \csc^2(\theta)$. we plug in cotangent and we get

$$\left(-\frac{3}{7}\right)^2 + 1 = \csc^2(\theta),$$

simplify the term on the left to get

$$\frac{9}{49} + 1 = \csc^2(\theta),$$

further simplify to get

$$\frac{58}{49} = \csc^2(\theta).$$

Taking the square root yields

$$\pm\sqrt{\frac{58}{49}} = \csc(\theta),$$

or in other words,

$$\csc(\theta) = \pm\frac{\sqrt{58}}{7}.$$

We must decide the sign of cosecant by the quadrant that θ lies in. Since we are told $\cot(\theta) = -\frac{3}{7}$, we know that θ must lie in either quadrant II or quadrant IV (see Table 5 on pg. 129). Similarly, since we are told that $\cos(\theta) > 0$, we know that θ must lie in either quadrant I or quadrant IV. Therefore it must be the case that θ lies in quadrant IV. The table also tells us that $\csc(\theta)$ is **negative** when θ is in quadrant IV. Therefore we must take the value

$$\csc(\theta) = -\frac{\sqrt{58}}{7}.$$

Now since $\csc(\theta) = \frac{1}{\sin(\theta)}$ we see that

$$\sin(\theta) = \frac{1}{\csc(\theta)} = \frac{1}{-\frac{\sqrt{58}}{7}} = -\frac{7}{\sqrt{58}},$$

and

$$\tan(\theta) = \frac{1}{\cot(\theta)} = -\frac{7}{3}.$$

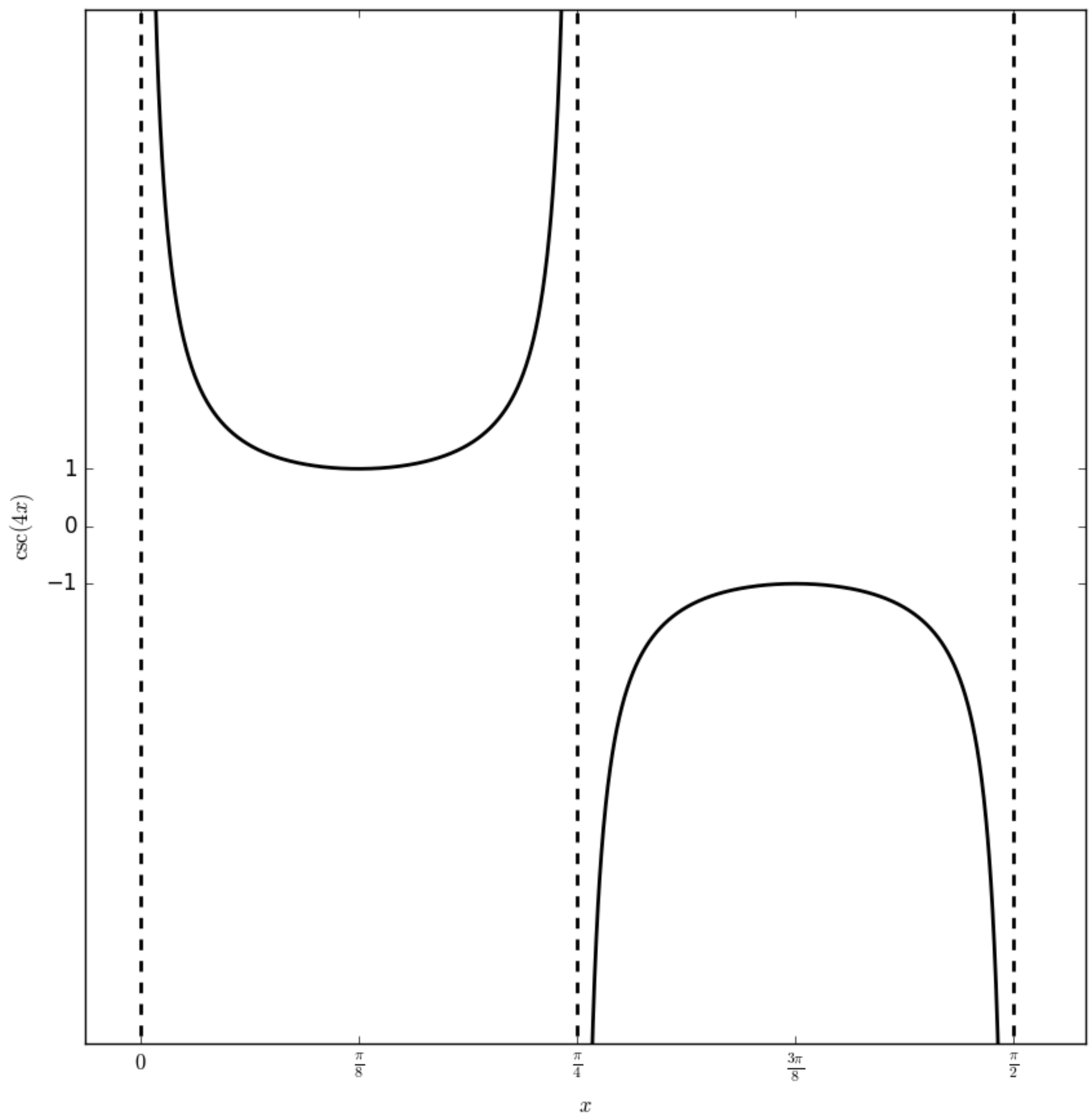
It remains to find the values of $\cos(\theta)$ and $\sec(\theta)$. To find cosine, recall that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, and so solving for $\cos(\theta)$ yields

$$\cos(\theta) = \frac{\sin(\theta)}{\tan(\theta)} = \frac{-\frac{7}{\sqrt{58}}}{-\frac{7}{3}} = \frac{3}{\sqrt{58}},$$

and finally

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\sqrt{58}}{3}.$$

8. (10 points) Graph $y = \csc(4x)$.
Solution: From the formula, we see that the only thing affected are the anchor points – it affects them by division by 4. We will start with the anchor points $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ and divide them by 4 to get the anchor points $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{2\pi}{4} = \frac{\pi}{2}$. Thus the graph looks like this:



NOTE: *Some people used the anchor points* $-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ *instead of those used above. That would yield a similar looking picture (the left piece will be the negative one in that one) and was accepted for full credit.*

9. (10 points) Graph $y = 4\sin\left(2x - \frac{\pi}{2}\right) + 1$.

Solution: Summarize the information from the equation:

Amplitude	4
Period change	divide by 2
Vertical shift	up 1
Phase shift	right by $\frac{\pi}{4}$

Taking the anchor points $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ and dividing them by 2 yields $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$. Shifting them right by $\frac{\pi}{4}$ yields the anchor points $\frac{\pi}{4}, \frac{2\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{4}, \frac{4\pi}{4} = \pi, \frac{5\pi}{4}$. Thus we get the following picture: