MATH 3304 Summer 2015
Homework 5 Name: ________________________

We have been studying differential equations of first and second order. This homework will introduce you to a related theory – difference equations. Difference equations is a theory where the "derivative" is replaced by the so-called "difference" operator Δ which is given by the formula

$$\Delta f(x) = f(x+1) - f(x).$$

1. (*2 points*) Let $f(x) = x^2 + 3x$. Calculate $\Delta f(x)$.

Solution: By the formula, we compute

$$\Delta f(x) = [(x+1)^2 + 3(x+1)] - [x^2 + 3x] = 2x + 4.$$

Note: From this we see that the difference operators do not obey the traditional "power rule"

$$\frac{d}{dx}x^n = nx^{n-1}.$$

A familiar power rule for the difference operator requires a different definition for "x^n".

The Δ operator has many properties similar to the derivative, but slightly different. For example, a product rule for Δ is given by

$$\Delta[f(x)g(x)] = g(x+1)\Delta[f(x)] + f(x)\Delta[g(x)].$$

Very simply, a "difference equation" is the same thing as a differential equation, except instead of derivatives we use the Δ operator in its place. Let α be a constant. We define the "discrete exponential function" by the difference equation $\Delta y(x) = \alpha y(x); y(0) = 1$.

2. (*3 points*) Solve the difference equation $\Delta y(x) = \alpha y(x)$ by rewriting the left-hand-side using the definition of Δ and then solving the resulting equation for $y(x+1)$. Use this equation and the initial condition to determine values of the solution at $x = 1, 2, 3, 4$.

Solution: Using the definition of Δ, we see

$$y(x+1) - y(x) = \alpha y(x)$$

or

$$y(x+1) = (\alpha+1)y(x).$$

Using our initial condition, we plug in $x = 0$ to see

$$y(1) = (\alpha+1)y(0) = \alpha+1.$$

Now we use this value and plug in $x = 1$ to see

$$y(2) = (\alpha+1)y(1) = (\alpha+1)^2,$$

plugging in $x = 2$ yields

$$y(3) = (\alpha+1)y(2) = (\alpha+1)^3,$$

and plugging in $x = 3$ yields

$$y(4) = (\alpha+1)y(3) = (\alpha+1)^4.$$

Note: in general, $y(n) = (\alpha+1)^n$ for all $n = 0, \pm1, \pm2$ if and only if $\alpha \neq -1$.

A popular notation for the solution found in Problem 2 is $e_\alpha(t)$; it obeys the formulas $\Delta e_\alpha(t) = \alpha e_\alpha(t)$ and $\Delta^2 e_\alpha(t) = \alpha^2 e_\alpha(t)$. With this information we can solve some second order homogeneous difference equations with constant coefficients by letting $y(t) = e_r(t)$, plugging it into the difference equation, and solving for r (just as we do for differential equations).

3. (*3 points*) Find the general solution of the second order difference equation

$$\Delta^2 y(t) - 4\Delta y(t) + 3y(t) = 0.$$

Solution: Plugging in the formulas above yields

$$(r^2 - 4r + 3)e_r(t) = 0.$$

If we assume that $e_r(t) \neq 0$ (we can check this when we find the solution), then we can divide by $e_r(t)$ to get

$$r^2 - 4r + 3 = 0,$$

which yields roots $r = 1, 3$. Therefore the general solution is

$$y(t) = c_1 e_1(t) + c_2 e_3(t).$$