MATH 3304 - EXAM 3 SUMMER 2015
SOLUTION

Friday 17 July 2015
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Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (28 points) Solve the following differential equation:

$$y'' + 3y' + 2y = \delta(t-1); y(0) = 0, y'(0) = 0.$$

Solution: Take the Laplace transform to get

$$(s^2\mathscr{L}\{y\} - sy(0) - y'(0)) + 3(s\mathscr{L}\{y\} - y(0)) + 2\mathscr{L}\{y\} = e^{-s}.$$

Applying initial conditions yields

$$(s^2 + 3s + 2)\mathscr{L}\{y\} = e^{-s},$$

and solving for $\mathscr{L}\{y\}$ gives us

$$\mathscr{L}\{y\} = \frac{e^{-s}}{s^2 + 3s + 2}.$$

Since

$$\frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)}$$

we use partial fractions:

$$\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1},$$

hence

$$1 = (A+B)s + (A+2B)$$

yielding the system

$$\begin{cases} A + B &= 0 \\ A + 2B &= 1, \end{cases}$$

which has solution $A = -1$ and $B = 1$. Hence

$$\mathscr{L}\{y\} = -\frac{e^{-s}}{s+2} + \frac{e^{-s}}{s+1}.$$

Take the inverse Laplace transform to get (note that the function $\theta(t-1)$ on Wolframalpha corresponds to our function $u_1(t)$)

$$\begin{aligned} y(t) &= -\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s+2}\right\}(t) + \mathscr{L}^{-1}\left\{\frac{e^{-s}}{s+1}\right\}(t) \\ &= -u_1(t)e^{-2t+2} + u_1(t)e^{1-t} \end{aligned}$$

2. (28 points) Solve the following integral equation:

$$y(t) + \int_0^t y(t-\tau)\tau d\tau = 4.$$

Solution: Recognize that the integral in this equation can be written as the convolution $(y * f)(t)$ where $f(t) = t$ (hence $\mathscr{L}\{f\} = \frac{1}{s^2}$), so we can rewrite the integral equation as

$$y(t) + (y * f)(t) = 4.$$

Take the Laplace transform, using the convolution theorem on the second term on the left-hand-side to get

$$\mathscr{L}\{y\} + \mathscr{L}\{y\}\mathscr{L}\{f\} = \mathscr{L}\{4\},$$

which simplifies to

$$\left(1 + \frac{1}{s^2}\right)\mathscr{L}\{y\} = \frac{4}{s}.$$

Since $1 + \frac{1}{s^2} = \frac{s^2+1}{s^2}$ we may solve for $\mathscr{L}\{y\}$ to get

$$\mathscr{L}\{y\} = \frac{4}{s}\frac{s^2}{s^2+1} = \frac{4s}{s^2+1}.$$

Take the inverse Laplace transform to get

$$y(t) = 4\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = 4\cos(t).$$

3. (28 points) Complete the following two parts.

(a) (20 points) Solve the following differential equation:

$$y''(t) + 4y(t) = f(t); y(0) = 0, y'(0) = 0, f(t) = \begin{cases} 0 & ; 0 \le t < 10 \\ 1 & ; t \ge 10 \end{cases}.$$

Solution: We must express f using step functions: $f(t) = u_{10}(t)$. Thus the differential equation becomes

$$y''(t) + 4y(t) = u_{10}(t).$$

Note that $\mathscr{L}\{f\} = \dfrac{e^{-10s}}{s}$. Take the Laplace transform of the ODE to get

$$(s^2\mathscr{L}\{y\} - sy(0) - y'(0)) + 4\mathscr{L}\{y\} = \frac{e^{-10s}}{s}.$$

Apply the initial conditions to this equation to get

$$(s^2 + 4)\mathscr{L}\{y\} = \frac{e^{-10s}}{s}.$$

Solving for $\mathscr{L}\{y\}$ yields

$$\mathscr{L}\{y\} = \frac{e^{-10s}}{s(s^2 + 4)}.$$

We will use partial fractions to simplify $\dfrac{1}{s(s^2 + 4)}$:

$$\frac{1}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4},$$

so

$$1 = (A + B)s^2 + Cs + 4A.$$

This yields the system

$$\begin{cases} A + B = 0 \\ C = 0 \\ 4A = 1, \end{cases}$$

which has solution $A = \frac{1}{4}, B = -\frac{1}{4}, C = 0$. Hence

$$\mathscr{L}\{y\} = \frac{1}{4}\frac{1}{s} - \frac{1}{4}\frac{s}{s^2 + 4}.$$

Taking the inverse Laplace transform yields

$$y(t) = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{e^{-10s}}{s}\right\} - \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{se^{-10s}}{s^2 + 4}\right\} = \frac{1}{4}u_{10}(t) - \frac{1}{4}u_{10}(t)\cos(2(t - 10)).$$

note: Wolfram's alpha solution of $\frac{1}{2}u_{10}(t)\sin^2(10 - t)$ is equivalent to our solution (look under "alternate forms").

(b) (8 points) Use the solution $y(t)$ you found above to compute the value of $y(2)$ and the value of $y(11)$.

Solution: Compute

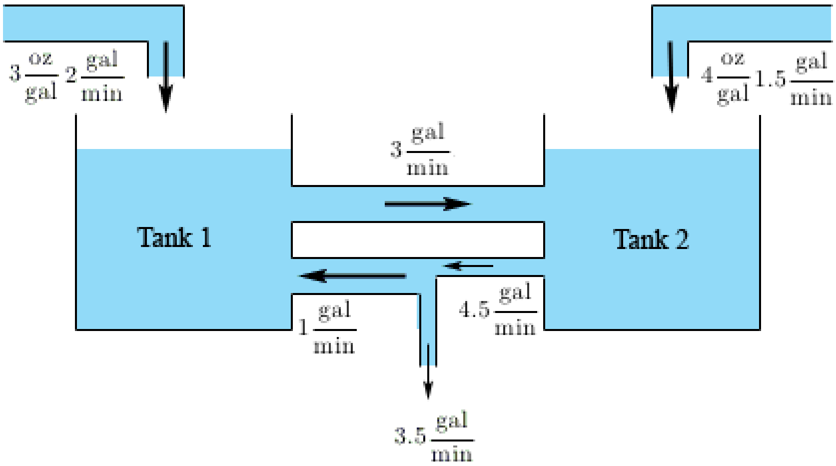
$$y(2) = \frac{1}{4}u_{10}(2) - \frac{1}{4}u_{10}(2)\cos(2(2 - 10)) = 0,$$

because of the definition of $u_{10}(t)$. Compute

$$\begin{aligned} y(11) &= \frac{1}{4}u_{10}(11) - \frac{1}{4}u_{10}(11)\cos(2(11-10)) \\ &= \frac{1}{4} - \frac{1}{4}\cos(2). \end{aligned}$$

4. (28 points) Consider two tanks filled with a solution of water and salt. Assume throughout that the solution is well-mixed. Tank 1 initially contains 10 gallons of water and 12 oz of salt, and Tank 2 initially contains 5 gallons of water and 4 oz of salt. Water containing $3\frac{\text{oz}}{\text{gal}}$ of salt flows into Tank 1 at a rate of $2\frac{\text{gal}}{\text{min}}$. The mixture flows from Tank 1 to Tank 2 at a rate of $3\frac{\text{gal}}{\text{min}}$. Water containing $4\frac{\text{oz}}{\text{gal}}$ of salt also flows into Tank 2 at a rate of $1.5\frac{\text{gal}}{\text{min}}$ (from the outside). The mixture drains from Tank 2 at a rate of $4.5\frac{\text{gal}}{\text{min}}$, of which some flows into Tank 1 at a rate of $1\frac{\text{gal}}{\text{min}}$, while the remainder leaves the system.

Set up but **do not solve** a system of differential equations that models the amount of salt in each tank at time t.



Solution: Let $Q_1(t)$ be the amount of salt in Tank 1 and $Q_2(t)$ be the amount of salt in Tank 2. Thus we have

$$\begin{cases} Q_1'(t) = \left(2\frac{\text{gal}}{\text{min}}\right)\left(3\frac{\text{oz}}{\text{gal}}\right) + \left(\frac{Q_2(t)}{5}\frac{\text{oz}}{\text{gal}}\right)\left(1\frac{\text{gal}}{\text{min}}\right) - \left(3\frac{\text{gal}}{\text{min}}\right)\left(\frac{Q_1(t)}{10}\frac{\text{oz}}{\text{gal}}\right) \\ Q_2'(t) = \left(1.5\frac{\text{gal}}{\text{min}}\right)\left(4\frac{\text{oz}}{\text{gal}}\right) + \left(\frac{3\text{gal}}{\text{min}}\right)\left(\frac{Q_1(t)}{10}\frac{\text{oz}}{\text{gal}}\right) - \left(\frac{4.5\text{gal}}{\text{min}}\right)\left(\frac{Q_2(t)}{5}\frac{\text{oz}}{\text{gal}}\right), \end{cases}$$

or simplified

$$\begin{cases} Q_1'(t) = -\frac{3}{10}Q_1(t) + \frac{1}{5}Q_2(t) + 6; Q_1(0) = 10 \\ Q_2'(t) = \frac{3}{10}Q_1(t) - \frac{4.5}{5}Q_2(t) + 6; Q_2(0) = 5 \end{cases}$$

5. (28 points) Solve the following system of two equations by transforming it into a second order differential equation:

$$\begin{cases} y_1'(t) = -2y_1(t) + y_2(t) & ; y_1(0) = 1 \\ y_2'(t) = 4y_1(t) + y_2(t) & ; y_2(0) = 1 \end{cases}$$

Solution: By the first equation in the system,

$$y_2 = y_1' + 2y_1$$

and we may compute

$$y_2' = y_1'' + 2y_1'.$$

Plug these formulas into the second equation of the system to get

$$y_1'' + 2y_1' = 4y_1 + y_1' + 2y_1$$

and simplify to get

$$y_1'' + y_1' - 6y_1 = 0.$$

This is a 2nd order ODE with constant coefficients – we solve it by looking at the characteristic equation

$$r^2 + r - 6 = 0$$

which factors to

$$(r+3)(r-2) = 0.$$

Thus we see

$$y_1(t) = c_1 e^{-3t} + c_2 e^{2t}.$$

Applying the initial condition to this yields

$$(*) \qquad 1 = y(0) = c_1 + c_2.$$

Differentiate the formula for y_1 to get

$$y_1'(t) = -3c_1 e^{-3t} + 2c_2 e^{2t}.$$

Now back substitute our formulas for y_1 and y_1' into the formula defining y_2 to see

$$\begin{aligned} y_2 &= (-3c_1 e^{-3t} + 2c_2 e^{2t}) + 2(c_1 e^{-3t} + c_2 e^{2t}) \\ &= -c_1 e^{-3t} + 4c_2 e^{2t}. \end{aligned}$$

Now apply the initial condition for y_2 to get

$$(**) \qquad 1 = y_2(0) = -c_1 + 4c_1.$$

Equations $(*)$ and $(**)$ yield the following system of equations to determine c_1 and c_2:

$$\begin{cases} c_1 + c_2 &= 1 \\ -c_1 + 4c_2 &= 1. \end{cases}$$

This system has solution $c_1 = \dfrac{3}{5}$ and $c_2 + \dfrac{2}{5}$. Therefore we have shown that the solution of the system of equations is

$$\begin{cases} y_1(t) = \dfrac{3}{5}e^{-3t} + \dfrac{2}{5}e^{2t} \\ y_2(t) = -\dfrac{3}{5}e^{-3t} + \dfrac{8}{5}e^{2t}. \end{cases}$$

Table of Laplace Transforms

$f(t) = \mathscr{L}^{-1}\{F(s)\}$	$F(s) = \mathscr{L}\{f(t)\}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$u_c(t)f(t)$	$e^{-cs}\mathscr{L}\{f(t+c)\}(s)$
$e^{ct}f(t)$	$F(s-c)$
$(f*g)(t)$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$