MATH 3304 - EXAM 1 SUMMER 2015
SOLUTIONS

Friday 19 June 2015
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (10 points) Circle **T** for true or **F** for false.

   (a) (2 points) **T** (**F**) The Existence and Uniqueness Theorem for the linear differential equation $y' + py = g$ requires that p and g be differentiable functions.
   Explanation: the theorem only requires continuity, not differentiability.

   (b) (2 points) **T** (**F**) The equation $\frac{dy}{dx} = 3x^2 y$ is an autonomous differential equation.
   Explanation: autonomous equations are of the form $y' = h(y)$, not $y' = h(y)g(x)$.

   (c) (2 points) (**T**) **F** The function $y(x) = \arcsin(x)$ is a solution of the differential equation

   $$x\sqrt{1-x^2}\frac{dy}{dx} - \sin(y) = 0.$$

   Explanation: we know that $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$. Plug this into the left hand side of the differential equation to see

   $$x\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}} - \sin(\arcsin(x)) = x - x = 0,$$

   proving that $y(x) = \arcsin(x)$ is a solution.

   (d) (2 points) **T** (**F**) The Existence and Uniqueness Theorem for nonlinear ODE's tells us that all nonlinear ODE's have unique solutions.
   Explanation: we saw an example in class of a nonlinear ODE with nonunique solutions (the equation $y' = \sqrt[3]{y}$).

   (e) (2 points) (**T**) **F** Let f be a continuous function. The fundamental theorem of calculus tells us that $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.
   Explanation: this is what the theorem says.

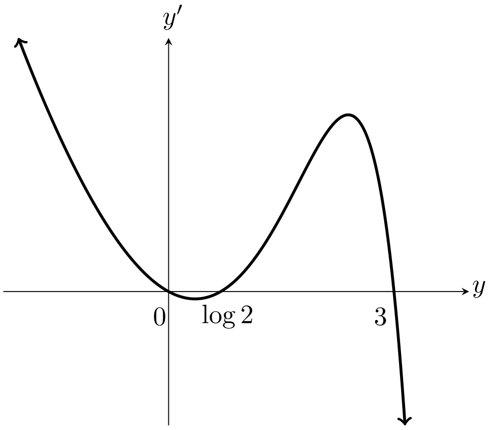
2. (17 points) Fill out the following table. If the equation is nonlinear, circle the term that makes it nonlinear. **Do not attempt to solve them!**

Differential equation	Order	Linear? Yes or no.
$y'(x) + 2y(x) = 6$	1	yes
$\boxed{\sqrt{y}} + 2ty' = 17e^{2t}$	1	no
$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$	2	yes
$\frac{d}{dx}\left[2x\frac{dy}{dx}\right] + y = 0$	2	yes
$\frac{d^{17}y}{dx^{17}} + 4\sqrt{x}\frac{dy}{dx} + \log(\sin(x))y = \boxed{2t\log(\cos(y))}$	17	no
$\boxed{2yy'} - 11y'' + e^{19t}y = 42$	2	no
$\frac{d^3\psi}{d\xi^3} + \frac{d^2\psi}{d\xi^2}\sqrt{\log(\xi)} = \xi^5\psi$	3	yes

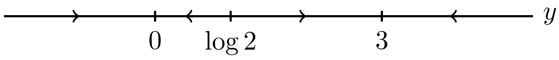
3. (26 points) Consider the differential equation $y' = (y^2 - 3y)(2 - e^y)$.

(a) (7 points) Find the equilibrium solutions.
Solution: An equilibrium solution is of the form $y(t) = \alpha$ for a constant α. Clearly $y'(t) = 0$. Plug this into the differential equation to get $0 = (y^2 - 3y)(2 - e^y)$ which has the solutions $y = 0, 3, \log 2$. These are the equilibrium solutions.

(b) (6 points) Draw the phase line for this equation (**SHOW YOUR WORK**).
Solution: First we plot y' versus y, to do this plot the zeros found in part (a) (realize that $\log(2)$ must be between 0 and 1 because $\log(1) = 0$, $\log(e) = 1$, and $e = 2.71\ldots$ are well known.) The standard technique to finish the plot is to pick a "test point" in each interval to figure out if the graph is above or below the x-axis and then fill in the curve.



From this graph it is clear that y is increasing on $(-\infty, 0)$, decreasing on $(0, \log 2)$, increasing on $(\log 2, 3)$, and decreasing on $(3, \infty)$. Thus we get the following phase line:



(c) (7 points) Classify each equilibrium as stable, unstable, or semistable.
Solution: From the phase line it is clear that 0 is a stable equilibrium, $\log 2$ is an unstable equilibrium, and 3 is a stable equilibrium.

(d) (6 points) If the initial condition $y(0) = 1$ is applied, what is the limit as $t \to \infty$ of the solution? Justify your answer.
Solution: This initial condition places us at $y = 1$. Our phase line describes what happens to the function $y(x)$ for various values of y. Looking at the phase line, we know that 1 belongs between $\log 2$ and 3 and hence the solution will tend toward the equilibrium solution 3 in the limit.

4. (25 points) Solve the initial value problem $ty'(t) + 17y(t) = \frac{\cos t}{t^{16}}; y(\pi) = 1$ for $t > 0$.

Solution: This is a first order linear ODE, but it is not separable. Therefore we will proceed via the method of integrating factors. First we need to divide by t to put the equation into standard form:

$$y'(t) + \frac{17}{t}y(t) = \frac{\cos t}{t^{17}}; y(\pi) = 1.$$

Hence our integrating factor is

$$\mu(t) = \exp\left(\int \frac{17}{t}dt\right) = e^{17\log t} = e^{\log(t^{17})} = t^{17}.$$

Multiply by μ and factor on the left side to get

$$(t^{17}y(t))' = \cos(t).$$

Integration with respect to t on both sides and the fundamental theorem of calculus yields

$$t^{17}y(t) = \sin(t) + C,$$

hence

$$y(t) = \frac{\sin(t)}{t^{17}} + \frac{C}{t^{17}}.$$

Now we will compute C from the initial condition:

$$1 = y(\pi) = \frac{\sin(\pi)}{\pi^{17}} + \frac{C}{\pi^{17}} = \frac{C}{\pi^{17}},$$

hence $C = \pi^{17}$. We now see the solution of the initial value problem is

$$y(t) = \frac{\sin(t) + \pi^{17}}{t^{17}}.$$

5. (17 points) Solve the differential equation $y'' - 3y' - 11y = 0$.
Solution: If we assume that $y(t) = e^{rt}$ and plug this into the differential equation, we get

$$(r^2 - 3r - 11)e^{rt} = 0.$$

Since e^{rt} never equals zero, we may divide and we are left with the algebra problem

$$r^2 - 3r - 11 = 0.$$

This polynomial does not factor nicely, so we will solve it using the quadratic formula:

$$r = \frac{3 \pm \sqrt{9 - 4(1)(-11)}}{2} = \frac{3 \pm \sqrt{53}}{2}.$$

Therefore we see that the solution of the differential equation is

$$y(t) = c_1 e^{\left(\frac{3+\sqrt{53}}{2}\right)t} + c_2 e^{\left(\frac{3-\sqrt{53}}{2}\right)t}.$$

6. (20 points) Determine the largest interval on which a unique solution of the following initial value problem exists:

$$(t^2+6t+9)y' + \frac{1}{t-2}y = t^5; y(7) = 15.$$

Solution: The existence and uniqueness theorem for linear first order differential equations applies to the standard first order linear ODE $y' + py = g$ and only requires that p and g be continuous in an interval. We will find the points of discontinuity of p and g and use that and the initial condition to determine the interval of validity. First put the ODE in standard form by dividing by t^2+6t+9:

$$y' + \frac{1}{(t-2)(t^2+6t+9)}y = \frac{t^5}{t^2+6t+9}.$$

We see that we have $p(t) = \dfrac{1}{(t-2)(t^2+6t+9)}$ and $g(t) = \dfrac{t^5}{t^2+6t+9}$. Clearly $p(t)$ has a discontinuity whenever $t-2=0$ (hence $t=2$) and both $p(t)$ and $g(t)$ have discontinuity when or $t^2+6t+9=0$ (factors to $(t+3)^3=0$, hence $t=-3$). Hence there are three possible choices for intervals: $(-\infty,-3),(-3,2)$, or $(2,\infty)$. The initial condition $y(7)=15$ requires us to use $t=7$, and hence we see that the solution exists on the interval $(2,\infty)$.

7. (25 points) A 3000 gallon talk initially contains 500 gallons of water that has 5 pounds of salt dissolved in it. Water enters the tank at a rate of $7\frac{\text{gal}}{\text{hr}}$ with a salt concentration of $\frac{1+\sin(t)}{\log(t+2)}\frac{\text{lb}}{\text{gal}}$. The (well-mixed) mixture is pumped out of the tank at a rate of $5\frac{\text{gal}}{\text{hr}}$.

(a) (5 points) How long does it take for the water in the tank to overflow?

Solution: Since the water is entering at the rate $7\frac{\text{gal}}{\text{hr}}$ and exiting at the rate $5\frac{\text{gal}}{\text{hr}}$ we see the total rate of change of the water is $7-5=2\frac{\text{gal}}{\text{hr}}$. The water initially contains 500 gallons and so as time proceeds will contain $500+2t$ gallons at time t (measured in hours). Since the tank has a capacity of 3000 gallons, we can see when the tank overflows by solving the equation

$$500+2t=3000,$$

yielding $t=1250$ hours until the tank overflows.

(b) (20 points) Set up, **but do not solve** an initial value problem that models the amount (in pounds) of salt in the tank.

Solution: Let $Q(t)$ denote the amount of salt in the tank at time t (measured in pounds). We will work from the idea that

$$\text{rate of change of } Q(t) = \text{rate of } Q(t) \text{ entering tank} - \text{rate of } Q(t) \text{ exiting tank.}$$

The left-hand-side of this is simply $\frac{dQ}{dt}$ – this has units $\frac{\text{lb salt}}{\text{hr}}$. The rate of $Q(t)$ entering the tank must have units $\frac{\text{lb salt}}{\text{hr}}$ and so we determine it by multiplication:

$$\left(7\frac{\text{gal}}{\text{hr}}\right)\left(\frac{1+\sin(t)}{\log(t+2)}\frac{\text{lb salt}}{\text{hr}}\right)=7\left(\frac{1+\sin(t)}{\log(t+2)}\right)\frac{\text{lb salt}}{\text{hr}}.$$

The rate of $Q(t)$ exiting the tank must be determined from the knowledge that there are $5\frac{\text{gal}}{\text{hr}}$ of solution exiting the tank and the physical specifications of the tank. To make the units correct, we must multiply by a quantity whose units are $\frac{\text{lb salt}}{\text{gal}}$. The numerator of this unit is simply $Q(t)$ itself – the number of pounds of salt in the tank. The denominator must be measured in gallons and so is the total amount of liquid in the tank at time t. As we found in part (a), this is $500+2t$. Thus we get the rate of $Q(t)$ exiting the tank as follows:

$$\left(5\frac{\text{gal}}{\text{hr}}\right)\left(\frac{Q(t)}{500+2t}\frac{\text{lb salt}}{\text{gal}}\right)=\frac{5Q(t)}{500+2t}\frac{\text{lb salt}}{\text{gal}}.$$

The initial condition will be given by $Q(0)=5$. Hence we have determined the initial value problem:

$$\frac{dQ}{dt}=7\left(\frac{1+\sin(t)}{\log(t+2)}\right)-\frac{5Q(t)}{500+2t}; Q(0)=5.$$