# $$\lim_{R\to\infty} R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{it}\log\left(1+\frac{1}{Re^{it}}\right)\mathrm{d}t=\pi$$

Tom Cuchta, David Grow, Nick Wintz

14 September 2016

Define

$$I(R) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{it}\log\left(1+\frac{1}{Re^{it}}\right)\mathrm{d}t.$$

Let $z = Re^{it}$, then $-\frac{i}{R}\mathrm{d}z = e^{it}\mathrm{d}t$. This yields

$$I(R) = -\frac{i}{R}\int_{-iR}^{iR}\log\left(1+\frac{1}{z}\right)\mathrm{d}z = -\frac{i}{R}\int_{-iR}^{iR}\log(z+1)-\log(z)\mathrm{d}z.$$

Since

$$\int \log(z)\mathrm{d}z = z\log(z) - z + C,$$

and

$$\int \log(z+1)\mathrm{d}z = (z+1)\log(z+1) - z + C,$$

we see

$$\begin{aligned}
I(R) &= -\frac{i}{R}\left[[(z+1)\log(z+1)-z]-[z\log(z)-z]\Big|_{-iR}^{iR}\right.\\
&= -\frac{i}{R}\left[(z+1)\log(z+1)-z\log(z)\Big|_{-iR}^{iR}\right.\\
&= -\frac{i}{R}\left[z\log\left(1+\frac{1}{z}\right)+\log(z+1)\Big|_{-iR}^{iR}\right.\\
&= -\frac{i}{R}\left[iR\log\left(1+\frac{1}{iR}\right)+\log(iR+1)-(-iR)\log\left(1-\frac{1}{iR}\right)-\log(1-iR)\right]\\
&= -\frac{i}{R}\left[iR\log\left(1-\frac{i}{R}\right)+\log(iR+1)+iR\log\left(1+\frac{i}{R}\right)-\log(1-iR)\right]\\
&= \log\left(1-\frac{i}{R}\right)-\frac{i}{R}\log(iR+1)+\log\left(1+\frac{i}{R}\right)+\frac{i}{R}\log(1-iR)
\end{aligned}$$

It is simple trigonometry to put $1+iR$, $1-iR$, $1-\frac{i}{R}$, and $1+\frac{i}{R}$ into their polar forms:

$$1+iR=\sqrt{R^2+1}e^{i\arctan(R)},$$

$$1-iR=\sqrt{R^2+1}e^{-i\arctan(R)},$$

$$1-\frac{i}{R}=\sqrt{\frac{1}{R^2}+1}\exp\left(-i\arctan\left(\frac{1}{R}\right)\right),$$

and

$$1+\frac{i}{R}=\sqrt{\frac{1}{R^2}+1}\exp\left(i\arctan\left(\frac{1}{R}\right)\right).$$

Therefore using the formula $\log(Re^{i\theta})=\log(R)+i\theta$,

$$\log(1+iR)=\log\left(\sqrt{R^2+1}\right)+i\arctan(R)=\frac{\log(R^2+1)}{2}+i\arctan(R),$$

$$\log(1-iR)=\log\left(\sqrt{R^2+1}\right)-i\arctan(R)=\frac{\log(R^2+1)}{2}-i\arctan(R),$$

so,

$$-\frac{i}{R}\log(iR+1)=-\frac{i}{2R}\log(R^2+1)+\frac{1}{R}\arctan(R)$$

and

$$\frac{i}{R}\log(1-iR)=\frac{i}{2R}\log(R^2+1)+\frac{1}{R}\arctan(R).$$

Also,

$$\log\left(1-\frac{i}{R}\right)=\frac{1}{2}\log\left(\frac{1}{R^2}+1\right)-i\arctan\left(\frac{1}{R}\right),$$

and

$$\log\left(1+\frac{i}{R}\right)=\frac{1}{2}\log\left(\frac{1}{R^2}+1\right)+i\arctan\left(\frac{1}{R}\right).$$

Therefore

$$\begin{aligned}
I(R) &= \log\left(1-\frac{i}{R}\right)-\frac{i}{R}\log(iR+1)+\log\left(1+\frac{i}{R}\right)+\frac{i}{R}\log(1-iR)\\
&= \left[\frac{1}{2}\log\left(\frac{1}{R^2}+1\right)-i\arctan\left(\frac{1}{R}\right)\right]+\left[-\frac{i}{2R}\log(R^2+1)+\frac{1}{R}\arctan(R)\right]\\
&\quad+\left[\frac{1}{2}\log\left(\frac{1}{R^2}+1\right)+i\arctan\left(\frac{1}{R}\right)\right]+\left[\frac{i}{2R}\log(R^2+1)+\frac{1}{R}\arctan(R)\right]\\
&= \log\left(\frac{1}{R^2}+1\right)+\frac{2}{R}\arctan(R)
\end{aligned}$$

Now the limit we are interested in is $\lim_{R\to\infty} RI(R)$.

**Lemma 1.** $\lim_{R\to\infty} R\log\left(\frac{1}{R^2}+1\right)=0 \quad \square$

Therefore we see

$$\begin{aligned}
\lim_{R\to\infty} R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{it}\log\left(1+\frac{1}{Re^{it}}\right)\mathrm{d}t &= \lim_{R\to\infty} RI(R) \\
&= \lim_{R\to\infty} R\log\left(\frac{1}{R^2}+1\right)+2\arctan(R) \\
&= \lim_{R\to\infty} 2\arctan(R) \\
&= \frac{2\pi}{2} \\
&= \pi,
\end{aligned}$$

as was to be shown.