REDUCTION OF BLACK-SCHOLES EQUATION TO THE HEAT EQUATION

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A European call option is a security giving the holder the right, but not the obligation, to buy a single share of a specific stock at a specified price E on a specified future date T. E is called the exercise (or strike) price and T is called the maturity (or expiration) date of the option.

Let $V = (S,t)$ denote the value of the call option at time $t \in [0,T]$ if the current market value of the stock is S dollars per share. If r denotes the (constant) short-term riskless interest rate and σ denotes the "volatility parameter" of the stock's price per share, then Black and Scholes showed that V satisfies the PDE

$$V_t = rV - rSV_S - \frac{\sigma^2 S^2 V_{SS}}{2} \tag{1}$$

for $S \in (0,\infty)$ and $t \in (0,T)$, subject to the boundary conditions

$$\begin{cases} (2) \qquad V(0,t) = 0 \text{ and } \lim\limits_{S=\infty} \dfrac{V(S,t)}{S} \to 1 \\ (3) \qquad V(S,T) = \max\{S-E, 0\}. \end{cases}$$

We shall derive the formula (4) below for the solution to (1)-(2)-(3):

$$V(s,t) = SN(d_1) - Ee^{r(t-T)}N(d_2), \tag{4}$$

where

$$d_1 = \frac{\log(\frac{S}{E}) + \left[r + \frac{\sigma^2}{2}\right](T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\log(\frac{S}{E}) + \left[r - \frac{\sigma^2}{2}\right](T-t)}{\sigma\sqrt{T-t}},$$

and

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-p^2/2} dp.$$

Define the new variables

$$\begin{cases} x = \log(\frac{S}{E}) \longleftrightarrow e^x = \dfrac{S}{E} \longleftrightarrow S = Ee^x \\ \tau = T - t \longleftrightarrow t = T - \tau. \end{cases}$$

Since $S \in (0,\infty)$ we see that $x \in \mathbb{R}$ and since $t \in [0,T]$ we see that $\tau \in [0,T]$. Differentiating the definition of x with respect to S yields

$$\frac{dx}{dS} = \frac{1}{\frac{S}{E}} \frac{1}{E} = \frac{1}{S}.$$

Differentiating the definition of τ with respect to t yields

$$\frac{d\tau}{dt} = 0 - 1 = -1.$$

Define the function

$$v(x,\tau) = \frac{V(S,t)}{E}.$$

Now compute $\dfrac{\partial V}{\partial t}$ by the chain rule:

$$\begin{aligned}
V_t &= \frac{\partial V}{\partial t} \\
&= \frac{\partial}{\partial t} E v(x,\tau) \\
&= E\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + E\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} \\
&= 0 - E\frac{\partial v}{\partial \tau} \\
&= -Ev_\tau.
\end{aligned}$$

Now compute $\dfrac{\partial V}{\partial S}$ by the chain rule:

$$\begin{aligned}
V_S &= \frac{\partial V}{\partial S} \\
&= E\frac{\partial v}{\partial x}\frac{\partial x}{\partial S} + E\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial S} \\
&= \frac{E}{S}\frac{\partial v}{\partial x} + 0 \\
&= \frac{E}{S} v_x.
\end{aligned}$$

Now compute $\dfrac{\partial^2 V}{\partial S^2}$ by the chain rule:

$$\begin{aligned}
V_{SS} &= \frac{\partial}{\partial S}\frac{\partial V}{\partial S} \\
&= \frac{\partial}{\partial S}\left[\frac{E}{S} v_x\right] \\
&= -E\frac{1}{S^2} v_x + \frac{E}{S}\left[\frac{\partial v_x}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial v_x}{\partial \tau}\frac{\partial \tau}{\partial S}\right] \\
&= -\frac{E}{S^2} v_x + \frac{E}{S}\left[\frac{1}{S} v_{xx} + 0\right] \\
&= \frac{E}{S^2}\left[v_{xx} - v_x\right].
\end{aligned}$$

Therefore equation (1) becomes

$$-Ev_\tau = rEv - rS\frac{E}{S}v_x - \frac{\sigma^2 S^2 \frac{E}{S^2}[v_{xx} - v_x]}{2}$$

or equivalently

$$-Ev_\tau = Erv - Erv_x - \frac{1}{2}\sigma^2 E[v_{xx} - v_x]$$

or equivalently

$$v_\tau = \frac{\sigma^2}{2} v_{xx} + \left(r - \frac{\sigma^2}{2}\right) v_x - rv. \tag{5}$$

Now by definition of v, condition (3) implies

$$\begin{aligned} v(x,0) &= \frac{V(S,T)}{E} \\ &= \frac{1}{E}\max\{S-E,0\} \\ &= \frac{1}{E}\max\{Ee^x-E,0\} \\ &= \max\{e^x-1,0\}. \end{aligned}$$

Now define the function $u(x,\tau)$ by the formula

$$v(x,\tau)=u(x,\tau)e^{\alpha x+\beta\tau}\longleftrightarrow v(x,\tau)e^{-(\alpha x+\beta\tau)}=u(x,\tau),$$

for some as-of-yet undetermined constants α,β. Compute

$$\left\{\begin{aligned} v_\tau &= u_\tau e^{\alpha x+\beta\tau}+\beta u e^{\alpha x+\beta\tau} \\ &= e^{\alpha x+\beta\tau}(\beta u+u_\tau), \\ v_x &= u_x e^{\alpha x+\beta\tau}+\alpha u e^{\alpha x+\beta\tau} \\ &= e^{\alpha x+\beta\tau}(\alpha u+u_x), \\ v_{xx} &= (\alpha^2 u+\alpha u_x)e^{\alpha x+\beta\tau}+e^{\alpha x+\beta\tau}(\alpha u_x+u_{xx}) \\ &= e^{\alpha x+\beta\tau}\left(u_{xx}+2\alpha u_x+\alpha^2 u\right). \end{aligned}\right\}$$

Substitute these values into (5) to get

$$e^{\alpha x+\beta\tau}(\beta u+u_\tau)=\frac{\sigma^2}{2}\left[e^{\alpha x+\beta\tau}\left(u_{xx}+2\alpha u_x+\alpha^2 u\right)\right]+\left(r-\frac{\sigma^2}{2}\right)\left[e^{\alpha x+\beta\tau}(\alpha u+u_x)\right]-rue^{\alpha x+\beta\tau},$$

or equivalently

$$\beta u+u_\tau=\frac{\sigma^2}{2}\left(u_{xx}+2\alpha u_x+\alpha^2 u\right)+\left(r-\frac{\sigma^2}{2}\right)(\alpha u+u_x)-ru,$$

or equivalently

$$u_\tau=\frac{\sigma^2}{2}u_{xx}+\left(2\alpha\frac{\sigma^2}{2}+r-\frac{\sigma^2}{2}\right)u_x+\left(\alpha^2\frac{\sigma^2}{2}+\alpha r-\alpha\frac{\sigma^2}{2}-r-\beta\right)u,$$

or equivalently

$$(*)\qquad u_\tau=\frac{\sigma^2}{2}u_{xx}+\left(\alpha\sigma^2+r-\frac{\sigma^2}{2}\right)u_x+\left(\frac{\alpha(\alpha-1)\sigma^2}{2}-r(\alpha-1)-\beta\right)u.$$

We wish to make the terms u_x and u disappear, so we will force their coefficients to disappear, i.e. we write

$$\left\{\begin{aligned} &(\dagger) \qquad \alpha\sigma^2+r-\frac{\sigma^2}{2}=0 \\ &(\ddagger) \qquad \frac{\alpha(\alpha-1)\sigma^2}{2}-r(\alpha-1)-\beta=0. \end{aligned}\right.$$

If we write $k=\dfrac{2r}{\sigma^2}\longleftrightarrow r=\dfrac{k\sigma^2}{2}$ the equation ($\dagger$) implies

$$\alpha=\frac{\frac{\sigma^2}{2}-r}{\sigma^2}=\frac{1}{2}-\frac{r}{\sigma^2}=\frac{1}{2}(1-k)=-\frac{1}{2}(k-1).$$

And equation (‡) implies that

$$\begin{aligned}
\beta &= \frac{\alpha(\alpha-1)\sigma^2}{2} - r(\alpha-1) \\
&= -\frac{1}{4}(k-1)\left(-\frac{1}{2}(k-1)-1\right)\sigma^2 + \frac{\sigma^2 k}{2}\left(-\frac{1}{2}(k-1)-1\right) \\
&= \sigma^2\left[\frac{1}{8}(k-1)^2 + \frac{1}{4}(k-1) - \frac{k}{4}(k-1) - \frac{k}{2}\right] \\
&= \sigma^2\left[\frac{1}{8}(k^2-2k+1) + \frac{1}{4}(k-1) - \frac{k^2}{4} + \frac{k}{4} - \frac{k}{2}\right] \\
&= \sigma^2\left[k^2\left(\frac{1}{8}-\frac{1}{4}\right) + k\left(-\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-\frac{1}{2}\right) + \left(\frac{1}{8}-\frac{1}{4}\right)\right] \\
&= \sigma^2\left[-\frac{k^2}{8} - \frac{k}{4} - \frac{1}{8}\right] \\
&= -\frac{\sigma^2}{8}(k^2+2k+1) \\
&= -\frac{\sigma^2(k+1)^2}{8}.
\end{aligned}$$

Since we chose α and β to force the u_x and u terms of equation $(*)$ to disappear, we see now that the function u satisfies

$$u_\tau = \frac{\sigma^2}{2}u_{xx}.$$

We can specify the initial conditions by

$$\begin{cases}
v(x,\tau) &= u(x,\tau)e^{\alpha x+\beta\tau} \\
v(x,0) &= u(x,0)e^{\alpha x} = \max\{e^x-1,0\} \\
u(x,0) &= e^{-\alpha x}v(x,0) \\
&= e^{-\alpha x}\max\{e^x-1,0\} \\
&= \max\left\{e^{x(1-\alpha)} - e^{-\alpha x}, 0\right\} \\
&= \max\left\{e^{x(1+\frac{1}{2}(k-1))} - e^{\frac{1}{2}(k-1)x}, 0\right\} \\
&= \max\left\{e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0\right\}.
\end{cases}$$

Therefore we have reduced equation (1) to the heat equation. We now solve this problem by noting the standard classical solution to our heat equation is

$$u(x,\tau) = \frac{1}{\sqrt{4\pi K\tau}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4K\tau}}\phi(y)dy$$

where $K = \dfrac{\sigma^2}{2}$ and $\phi(y) = \max\left\{e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y}, 0\right\}$, yielding, since $y<0$ iff $e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y} < 0$ because exp is increasing,

$$\begin{aligned}
u(x,\tau) &= \int_{-\infty}^{\infty}\max\left\{e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y}, 0\right\}\frac{e^{-\frac{(x-y)^2}{4K\tau}}}{\sqrt{2\pi\sigma^2\tau}}dy \\
&= \int_0^{\infty}\left[e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y}\right]\frac{e^{-\frac{(x-y)^2}{4K\tau}}}{\sqrt{2\pi\sigma^2\tau}}dy.
\end{aligned}$$

We will reduce this u to a cumulative normal distribution function. Define

$$z = \frac{y-x}{\sigma\sqrt{\tau}} \longleftrightarrow y = z\sigma\sqrt{\tau} + x$$

so that

$$dz = \frac{1}{\sigma\sqrt{\tau}}dy$$

and so $u(x,\tau)$ becomes

$$\begin{aligned} u(x,\tau) &= \int_0^\infty \left[e^{\frac{1}{2}(k+1)y} - e^{\frac{1}{2}(k-1)y}\right] \frac{e^{-\frac{(x-y)^2}{2\sigma^2\tau}}}{\sqrt{2\pi\sigma^2\tau}}dy \\ &= \int_{-\frac{x}{\sigma\sqrt{\tau}}}^\infty \left[e^{\frac{1}{2}(k+1)(z\sigma\sqrt{\tau}+x)} - e^{\frac{1}{2}(k-1)(z\sigma\sqrt{\tau}+x)}\right] \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz \\ &= \int_{-\frac{x}{\sigma\sqrt{\tau}}}^\infty e^{\frac{1}{2}(k+1)(z\sigma\sqrt{\tau}+x)} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz - \int_{-\frac{x}{\sigma\sqrt{\tau}}}^\infty e^{\frac{1}{2}(k-1)(z\sigma\sqrt{\tau}+x)} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz \\ &= I_1 - I_2, \end{aligned}$$

where

$$I_1 = e^{\frac{1}{2}(k+1)x} \int_{-\frac{x}{\sigma\sqrt{\tau}}}^\infty e^{\frac{1}{2}(k+1)z\sigma\sqrt{\tau}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz$$

and

$$I_2 = e^{\frac{1}{2}(k-1)x} \int_{-\frac{x}{\sigma\sqrt{\tau}}}^\infty e^{\frac{1}{2}(k-1)z\sigma\sqrt{\tau}+} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}dz.$$

Consider the exponent of the integrand of I_1, i.e.

$$-\frac{z^2}{2} + \frac{z\sigma\sqrt{\tau}(k+1)}{2}.$$

Let us complete the square of this polynomial by adding and subtracting the term $\frac{1}{8}\sigma^2\tau(k+1)^2$ because

$$\begin{aligned} -\frac{1}{2}\left(z - \frac{1}{2}\sigma\sqrt{\tau}(k+1)\right)^2 &= -\frac{1}{2}\left(z^2 - z\sigma\sqrt{\tau}(k+1) + \frac{1}{4}\sigma^2\tau(k+1)^2\right) \\ &= -\frac{z^2}{2} + \frac{z\sigma\sqrt{\tau}(k+1)}{2} - \frac{1}{8}\sigma^2\tau(k+1)^2. \end{aligned}$$

We have shown that

$$I_1 = e^{\frac{1}{2}(k+1)x} e^{\frac{1}{8}\sigma^2\tau(k+1)^2} \int_{-\frac{x}{\sigma\sqrt{\tau}}}^\infty e^{-\frac{1}{2}\left(z-\frac{1}{2}\sigma\sqrt{\tau}(k+1)\right)^2} \frac{dz}{\sqrt{2\pi}}.$$

Now define the variable $w = z - \frac{1}{2}\sigma\sqrt{\tau}(k+1)$ so that $dw = dz$ and since it is well known that

$$\lim_{d\to\infty} N(d) = 1,$$

we see that

$$1 - N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{p^2}{2}}dp - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-p^2}dp = \frac{1}{\sqrt{2\pi}} \int_d^\infty e^{-p^2}dp$$

and

$$\begin{aligned}
1-N(-d) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-p^2}dp - \frac{1}{\sqrt{2\pi}}\int_{-d}^{\infty} e^{-p^2}dp \\
&\overset{\rho=-p}{=} \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} e^{-\rho^2}d\rho - \frac{1}{\sqrt{2\pi}}\int_{d}^{-\infty} e^{-\rho^2}d\rho \\
&= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\rho^2}d\rho - \frac{1}{\sqrt{2\pi}}\int_{d}^{-\infty} e^{-\rho^2}d\rho \\
&= -\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-\rho^2}d\rho - \int_{d}^{-\infty} e^{-\rho^2}d\rho\right] \\
&= -\frac{1}{\sqrt{2\pi}}\int_{d}^{-\infty} e^{-\rho^2}d\rho \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\rho^2}d\rho \\
&= N(d).
\end{aligned}$$

Thus we get

$$\begin{aligned}
I_1 &= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}\sigma^2\tau(k+1)^2}\int_{-\frac{x}{\sigma\sqrt{\tau}}-\frac{1}{2}\sigma\sqrt{\tau}(k+1)}^{\infty} e^{-\frac{w^2}{2}}\frac{dw}{\sqrt{2\pi}} \\
&= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}\sigma^2\tau(k+1)^2}\int_{\frac{-x-\frac{1}{2}\sigma^2\tau(k+1)}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{w^2}{2}}\frac{dw}{\sqrt{2\pi}} \\
&= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}\sigma^2\tau(k+1)^2}\left[1-N\left(-\frac{x+\frac{1}{2}\sigma^2\tau(k+1)}{\sigma\sqrt{\tau}}\right)\right] \\
&= e^{\frac{1}{2}(k+1)x}e^{\frac{1}{8}\sigma^2\tau(k+1)^2}N\left(\frac{x+\frac{1}{2}\sigma^2\tau(k+1)}{\sigma\sqrt{\tau}}\right).
\end{aligned}$$

The same set of calculations with the cumulative normal distribution applied to I_2 yields the formula

$$I_2 = e^{\frac{1}{2}(k-1)x}e^{\frac{1}{8}\sigma^2\tau(k-1)^2}N\left(\frac{x+\frac{1}{2}\sigma^2\tau(k-1)}{\sigma\sqrt{\tau}}\right).$$

So now by back substitution we see that

$$\begin{aligned}
\frac{x+\frac{1}{2}\sigma^2\tau(k+1)}{\sigma\sqrt{\tau}} &= \frac{\log(\frac{S}{E})+\frac{1}{2}\sigma^2(T-t)(\frac{2r}{\sigma^2}+1)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(\frac{S}{E})+(T-t)(r+\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}} \\
&= d_1,
\end{aligned}$$

and

$$\begin{aligned}
\frac{x+\frac{1}{2}\sigma^2\tau(k-1)}{\sigma\sqrt{\tau}} &= \frac{\log(\frac{S}{E})+\frac{1}{2}\sigma^2(T-t)(\frac{2r}{\sigma^2}-1)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(\frac{S}{E})+\frac{1}{2}(T-t)(2r-\sigma^2)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(\frac{S}{E})+(T-t)\left(r-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \\
&= d_2.
\end{aligned}$$

So we have

$$\begin{aligned} u(x,\tau) &= I_1 + I_2 \\ &= e^{\frac{1}{2}(k+1)x} e^{\frac{1}{8}\sigma^2\tau(k+1)^2} N\left(d_1\right) + e^{\frac{1}{2}(k-1)x} e^{\frac{1}{8}\sigma^2\tau(k-1)^2} N\left(d_2\right) \end{aligned}$$

and so

$$\begin{aligned} v(x,\tau) &= e^{\alpha x+\beta\tau} u(x,\tau) \\ &= e^{-\frac{1}{2}(k-1)x-\frac{\sigma^2(k+1)^2}{8}\tau} \left[e^{\frac{1}{2}(k+1)x} e^{\frac{1}{8}\sigma^2\tau(k+1)^2} N\left(d_1\right) + e^{\frac{1}{2}(k-1)x} e^{\frac{1}{8}\sigma^2\tau(k-1)^2} N\left(d_2\right) \right] \\ &= e^x N(d_1) + e^{-\frac{\sigma^2\tau k}{2}} N(d_1). \end{aligned}$$

Therefore recalling $r = \dfrac{k\sigma^2}{2}$, we have shown that

$$\begin{aligned} V(S,t) &= Ev(x,\tau) \\ &= Ev\left(\log(\tfrac{S}{E}), T-t\right) \\ &= E\left[e^{\log(\frac{S}{E})} N(d_1) + e^{-\frac{\sigma^2(T-t)k}{2}} N(d_2) \right] \\ &= SN(d_1) + Ee^{-r(T-t)} N(d_2) \\ &= SN(d_1) + Ee^{r(t-T)} N(d_2), \end{aligned}$$

as was to be shown.