ZERO-DIVISOR GRAPHS OF LOCALIZATIONS AND MODULAR RINGS

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ABSTRACT. In this paper, we examine the algebraic properties of localizations of commutative rings and how localizations affect the zero-divisor graph's structure of modular rings. We also classify the zero-divisor graphs of modular rings with respect to both the diameter and girth of their resultant zero-divisor graphs.

1. INTRODUCTION

In [4], Beck introduced the concept of the graph of a ring. Beck's main purpose was to examine the colorings of commutative rings. The work was continued by D.D. Anderson and Naseer in [1]. In these two papers all ring elements were included in the graph. However, this paper only includes the non-zero zero-divisors as vertices in the graph, just as D.F. Anderson and Livingston introduced in [2]. The diameter and girth of these zero-divisor graphs, among other things, were examined by D.F. Anderson and Livingston in [2], by Mulay in [9], and by DeMeyer and Schneider in [6]. In [2] it was shown that all zero-divisor graphs of commutative rings must be connected with diameter less than or equal to three and girth three, four, or infinity.

Throughout, R will denote a commutative ring with unity. The *powerset* of R will be denoted $\mathcal{P}(R)$. In this paper we will only consider proper ideals of R. A *prime ideal* P of a commutative ring R is an ideal of R such that if $ab \in P$, then $a \in P$ or $b \in P$. A *zero-divisor* is an element $z \in R$ such that $zr = 0$ for some nonzero $r \in R$. The set of zero-divisors of R will be denoted by $Z(R)$, and $Z(R)^*$ $= Z(R)\backslash\{0\}$.

Let R be a commutative ring with unity. Let S be a multiplicatively closed subset of R. Define a binary relation $\sim$ on $R \times S$ by $(r, s) \sim (r', s')$ if and only if there exists $s^* \in R\backslash S$ such that $s^*(rs' - r's) = 0$. This relation is an equivalence relation. Define $\frac{r}{s} = (r, s)$. The localization of R at S is the set $R_S = \{[\frac{r}{s}] \mid r \in R$ and $s \in S\}$ together with two binary operations $+, \cdot : R_S \times R_S \to R_S$ defined by $[\frac{r}{s}] + [\frac{r'}{s'}] = [\frac{rs'+r's}{ss'}]$ and $[\frac{r}{s}] \cdot [\frac{r'}{s'}] = [\frac{rr'}{ss'}]$. By the usual convention, $\cdot$ is replaced by juxtaposition, that is, $x \cdot y = xy$. Note that R_S is a commutative ring with additive identity $[\frac{0}{1}]$ and unity $[\frac{1}{1}]$. Notice that the complement of a prime ideal is always multiplicatively closed. By convention, a localization $R_{R\backslash P}$, where P is a prime ideal, is denoted R_P.

The *zero-divisor graph* of R is denoted $\Gamma(R) = \{V, E\}$, where the vertices $V = \{z \mid z \in Z(R)^*\}$ and the edges $E = \{(z, z') \mid zz' = 0$ and $z, z' \in V\}$ (edges are sometimes denoted as (a, b) where a and b are vertices). A *path* in a graph is a sequence of vertices, $a_0, a_1, ..., a_n$ such that each adjacent pair, a_i, a_{i+1}, is a

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nonrepeated valid edge in the edge set; (a_n, a_0) may or may not be an edge. The *distance* between two vertices in a graph is a path with the least number of edges. The *diameter* of a graph G, denoted diam(G), is the largest distance between any two vertices. A *cycle* is a path $a_0, a_1, ..., a_n$ such that (a_n, a_0) is an edge. The *girth* of a graph G, denoted g(G), is the length of a cycle with the least number of edges. A graph has girth ∞ if it contains no cycles. A graph is *connected* if there exists a path between all vertices of the graph.

A reference for localizations can be found in [8], and a reference for graph theory can be found in [5]. We compare the zero-divisor graphs of R and R_P, the localization of R around P. We consider the localizations of modular rings, and show they are isomorphic to a particular $\mathbb{Z}_m$. We also completely classify the zero-divisor graphs of modular rings by diameter and girth.

2. $\Gamma(R)$ and $\Gamma(R_P)$

From the definition of $\sim$, an element $[\frac{r}{s}]$ of R_S is in the equivalence class of $[\frac{0}{1}]$ if and only if there exists an $s^* \in S$ such that $s^* r = 0$. Clearly, if $[\frac{r}{s}] = [\frac{0}{1}]$, then for any $s' \in S$ $[\frac{r}{s'}] = [\frac{0}{1}]$.

Lemma 1. *If* $[\frac{r}{s}] \in Z(R_P)$*, then for any* $\bar{s} \in R \backslash P$*,* $[\frac{r}{\bar{s}}] \in Z(R_P)$*.*

Proof. Assume $[\frac{r}{s}] \in Z(R_P)$. Then there exists $[\frac{r'}{s'}] \neq [\frac{0}{1}]$ such that $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$. So there exists $s^* \in R \backslash P$ such that $s^* r r' = 0$. Let $[\frac{r}{\bar{s}}] \in R_P$. Then since $s^* r r' = 0$, $[\frac{r}{\bar{s}}][\frac{r'}{s'}] = [\frac{0}{1}]$. Thus, $[\frac{r}{\bar{s}}] \in Z(R_P)$. □

Define the numerator function $n : R_S \to \mathcal{P}(R)$ by $n([\frac{r}{s}]) = \{r' \in R \mid [\frac{r}{s}] = [\frac{r'}{s'}]\}$.

Lemma 2. *If* $[\frac{r}{s}] \in Z(R_P)$*, then for every* $\hat{r} \in n([\frac{r}{s}])$*,* $\hat{r} \in P \cap Z(R)$*.*

Proof. Suppose $[\frac{r}{s}] \in Z(R_P)$. Let $[\frac{r'}{s'}] \neq [\frac{0}{1}]$ such that $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$. By definition, there exists $s^* \in R \backslash P$ such that $s^* r r' = 0$, so $r \in Z(R)$. We know for every $\hat{r} \in n([\frac{r}{s}])$, $[\frac{r}{s}] = [\frac{\hat{r}}{\hat{s}}]$ for some $\hat{s}$. So, suppose $\hat{r} \notin P$. Then we know that $s^* \hat{r} \notin P$, since P is a prime ideal. If $\bar{s} = s^* \hat{r}$, then $\bar{s} r' = 0$, which implies that $[\frac{r'}{s'}] = [\frac{0}{1}]$, a contradiction. Thus $\hat{r} \in P$. □

Definition 3. *The total quotient ring of* R*, denoted* $T(R)$*, is the localization* R_S *where* $S = R \backslash Z(R)$*.*

The following lemmas present some relations between elements in $T(R)$ and elements in R_P.

Lemma 4. *Assume* $Z(R) \subseteq P$*. Then the following are equivalent.*

i) $[\frac{r}{1}][\frac{r'}{1}] = [\frac{0}{1}]$ *in* $T(R)$*.*
ii) $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ *in* $T(R)$ *for all* $s, s' \in R \backslash Z(R)$*.*
iii) $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ *in* R_P *for all* $s, s' \in R \backslash P$*.*
iv) $rr' = 0$*.*

Proof. $(i \Rightarrow ii)$ Assume $[\frac{r}{1}][\frac{r'}{1}] = [\frac{0}{1}]$ in $T(R)$. Then there exists $s^* \in R \backslash Z(R)$ such that $s^* r r' = 0$. But $s^* \notin Z(R)$, so $rr' = 0$ which implies $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in $T(R)$ for any $s, s' \in R \backslash Z(R)$.

$(ii \Rightarrow iii)$ Assume $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in $T(R)$ for all $s, s' \in R \backslash Z(R)$. Then there exists $s^* \in R \backslash Z(R)$ such that $s^* r r' = 0$. Since $s^* \notin Z(R)$, we know $rr' = 0$. Therefore, $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in R_P for all $s, s' \in R \backslash P$.

($iii \Rightarrow iv$) Assume $[\frac{r}{s}][\frac{r'}{s'}] = [\frac{0}{1}]$ in R_P. Then there exists $s^* \in R \backslash P$ such that $s^* rr' = 0$. We know $s^* \notin Z(R)$ since $Z(R) \subseteq P$, so $rr' = 0$.

($iv \Rightarrow i$) Clear. □

The following lemma shows that a pair of equivalence classes that are equal in R_P are also equal in $T(R)$, under a particular condition.

Lemma 5. *Assume $Z(R) \subseteq P$. Then $[\frac{r}{s}] = [\frac{r'}{s'}]$ in R_P if and only if $[\frac{r}{s}] = [\frac{r'}{s'}]$ in $T(R)$.*

Proof. ($\Rightarrow$) Assume $[\frac{r}{s}] = [\frac{r'}{s'}]$ in R_P. Then there exists $s^* \in R \backslash P$ such that $s^*(rs' - r's) = 0$. Since $s^* \notin Z(R)$, we know $rs' - r's = 0$. Hence, $[\frac{r}{s}] = [\frac{r'}{s'}]$ in $T(R)$.

($\Leftarrow$) Assume $[\frac{r}{s}] = [\frac{r'}{s'}]$ in $T(R)$. Then there exists $s^* \in R \backslash Z(R)$ such that $s^*(rs' - r's) = 0$. Since $s^* \notin Z(R)$, we know $rs' - r's = 0$. Thus, $[\frac{r}{s}] = [\frac{r'}{s'}]$ in R_P. □

The next lemma establishes an expected result between zero-divisors of R and their respective canonical fractions.

Lemma 6. *Assume $Z(R) \subseteq P$. Let $r, r' \in Z(R)^*$. Then, $r = r'$ if and only if $[\frac{r}{1}] = [\frac{r'}{1}]$ in R_P.*

Proof. ($\Leftarrow$) Assume $[\frac{r}{1}] = [\frac{r'}{1}]$ in R_P. Then, there exists $s^* \in R \backslash P$ such that $s^*(r - r') = 0$. Since $s^* \notin Z(R)$, we know $r - r' = 0$, which implies $r = r'$.

($\Rightarrow$) Trivial. □

It would be convenient to be able to create a homomorphism between R and $T(R)$, and the next lemma establishes that under certain circumstances, the obvious candidate for a homomorphism will suffice.

Lemma 7. *If $R = Z(R) \cup U(R)$, then for every $[\frac{r}{s}] \in T(R)$, there exists $r' \in R$ such that $[\frac{r}{s}] = [\frac{r'}{1}]$.*

Proof. Consider $r' = s^{-1}r$. Then, for all $s^* \in R \backslash Z(R)$, we have $s^*(r - sr') = s^*(r - ss^{-1}r) = 0$, which implies $[\frac{r}{s}] = [\frac{r'}{1}]$. □

A generalized version of the following theorem was presented and proved as Theorem 2.2 in [3]. However, we will now present a different proof below.

Theorem 8. *If R is a commutative ring with identity such that $R = Z(R) \cup U(R)$, then $R \cong T(R)$.*

Proof. Consider the function $\phi : R \to T(R)$ defined by $\phi(r) = [\frac{r}{1}]$. Consider $[\frac{r}{1}], [\frac{r'}{1}] \in T(R)$ such that $[\frac{r}{1}] = [\frac{r'}{1}]$. Then, there exists $s^* \in R \backslash Z(R)$ such that $s^*(r - r') = 0$. But $s^* \notin Z(R)$, so $r - r' = 0$ implies $r = r'$, thus ϕ is injective. Consider any $[\frac{r}{s}] \in T(R)$. By Lemma 7, we know that $[\frac{r}{s}] = [\frac{r'}{1}]$ for some $r' \in R$. Thus, $\phi(r') = [\frac{r'}{1}] = [\frac{r}{s}]$, so ϕ is surjective. Let $r, r' \in R$. Then, $\phi(r + r') = [\frac{r+r'}{1}] = [\frac{r}{1}] + [\frac{r'}{1}] = \phi(r) + \phi(r')$ and $\phi(rr') = [\frac{rr'}{1}] = [\frac{r}{1}][\frac{r'}{1}] = \phi(r)\phi(r')$, so ϕ is an operation preserving function. Thus, $R \cong T(R)$. □

Note that since $R \cong T(R)$, it follows trivially that $\Gamma(R) \cong \Gamma(T(R))$ under the above conditions.

3. Localizations of $\mathbb{Z}_n$

Consider $\mathbb{Z}_n$, where $n{=}{p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$ and each p_i is prime. Since $\mathbb{Z}_n$ is a principal ideal ring, every ideal is generated by a single element. Notice that prime ideals in $\mathbb{Z}_n$ will be generated by prime factors of n. Thus, we will only be concerned only with localizations of $\mathbb{Z}_n$ around ideals of the form (p_i), where $1 \le i \le k$.

Lemma 9. *If* $[\frac{q{p_i}^{e_i}+r}{1}] \in \mathbb{Z}_{n(p_i)}$*, then* $[\frac{q{p_i}^{e_i}+r}{1}] = [\frac{r}{1}]$.

Proof. All elements in $\mathbb{Z}_n$ can be written as $q{p_i}^{e_i} + r$ for some $0 \le r < {p_i}^{e_i}$ by the division algorithm. Thus $[\frac{q{p_i}^{e_i}+r}{1}] = [\frac{r}{1}]$, since if $s^* \in \mathbb{Z}_n \backslash (p_i)$ such that $s^* = {p_1}^{e_1}...{p_{i-1}}^{e_{i-1}}{p_{i+1}}^{e_{i+1}}...{p_k}^{e_k}$, then $s^* q{p_i}^{e_i} = 0$ in $\mathbb{Z}_n$. □

The following lemma is in a similar spirit to Lemma 7. It establishes that we can use the canonical mapping between $\mathbb{Z}_n$ and prime ideal localizations of $\mathbb{Z}_n$ as the basis of an isomorphism.

Lemma 10. *If* $[\frac{r}{s}] \in \mathbb{Z}_{n(p_i)}$*, then* $[\frac{r}{s}] = [\frac{r'}{1}]$ *in* $\mathbb{Z}_{n(p_i)}$ *for some* $r' \in \mathbb{Z}_n$.

Proof. Let $[\frac{r}{s}] \in \mathbb{Z}_{n(p_i)}$. Since $s \in \mathbb{Z}_n \backslash (p_i)$, $\gcd(p_i^{e_i}, s) = 1$, and thus $({p_i}^{e_i}) \cong \mathbb{Z}_s$. So, $r \equiv m{p_i}^{e_i} (\text{mod } s)$ for some m. Therefore, there exists $r' = m'{p_i}^{e_i}$ for some $r', m' \in \mathbb{Z}_n$ such that $sr' \equiv r - mp_i^{e_i} (\text{mod } s)$. Hence $m{p_i}^{e_i} \equiv r - sr' (\text{mod } s)$. Then, there exists $\bar{s} \in \mathbb{Z}_n \backslash (p_i)$, namely $\bar{s} = {p_1}^{e_1}...{p_{i-1}}^{e_{i-1}}{p_{i+1}}^{e_{i+1}}...{p_k}^{e_k}$, such that $\bar{s}(r - sr') = \bar{s}(mp_i^{e_i} - sm'p_i^{e_i}) = 0$ in $\mathbb{Z}_n$ which, by definition, implies $[\frac{r}{s}] = [\frac{r'}{1}]$. □

In general, it is a difficult task to determine a ring that is isomorphic to a given localization. The following theorem yields a very simple isomorphism for localizations around prime ideals of modular rings.

Theorem 11. *Let* $n{=}{p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$*. Then* $\mathbb{Z}_{n(p_i)} \cong \mathbb{Z}_{{p_i}^{e_i}}$.

Proof. Consider $\phi : \mathbb{Z}_{{p_i}^{e_i}} \to \mathbb{Z}_{n(p_i)}$ defined by $\phi(r) = [\frac{r}{1}]$. Assume $\phi(r) = \phi(\bar{r})$. Then $[\frac{r}{1}] = [\frac{\bar{r}}{1}]$. By definition, there exists an $s^* \in \mathbb{Z}_n \backslash (p_i)$ such that $s^*(r - \bar{r}) = 0$. Thus, $p_i^{e_i}$ must divide $r - \bar{r}$, which is impossible unless $r = \bar{r}$. So, ϕ is injective.

Let $[\frac{r}{s}] \in \mathbb{Z}_{n(p_i)}$. Then by the previous lemma, we know $[\frac{r}{s}] = [\frac{r'}{1}]$ for some $r' \in \mathbb{Z}_n$. So, $r' \equiv m \ (\text{mod } n)$. Then, since $n = p_1^{e_1} \dots p_k^{e_k}$, $r' \equiv m \ (\text{mod } {p_i}^{e_i})$. Then, $\phi(r') = \phi(m) = [\frac{m}{1}]$, and since $r' \equiv m \ (\text{mod } n)$, this implies $[\frac{m}{1}] = [\frac{r'}{1}] = [\frac{r}{s}]$, and hence ϕ is surjective.

Take $r, \bar{r} \in \mathbb{Z}_{{p_i}^{e_i}}$. Then, $\phi(r + \bar{r}) = [\frac{r+\bar{r}}{1}] = [\frac{r}{1}] + [\frac{\bar{r}}{1}] = \phi(r) + \phi(\bar{r})$. Similarly, $\phi(r\bar{r}) = [\frac{r\bar{r}}{1}] = [\frac{r}{1}][\frac{\bar{r}}{1}] = \phi(r)\phi(\bar{r})$. □

Corollary 12. *Let* $n = {p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$*. Then* $\Gamma(\mathbb{Z}_{n(p_i)}) \cong \Gamma(\mathbb{Z}_{{p_i}^{e_i}})$.

The isomorphism established above inspires a question about whether localizations around non-prime ideals have other nice isomorphisms. It also hints at the possibility of an unexplored generalization of the theorem to localizations of direct product decompositions, since a modular ring can be decomposed into a direct product of powers of primes.

4. Classification of $\Gamma(\mathbb{Z}_n)$

Theorem 13. *The following table holds true:*

Factorization of n	*Diameter*	*Girth*
p; p is prime	-	-
2^2	*0*	∞
3^2	*1*	∞
p^2*; p is prime and* $p > 3$	*1*	*3*
2^3*, or* $2p$*; p odd prime*	*2*	∞
pq; p,q, distinct odd primes	*2*	*4*
p^m*; p is prime,* $m > 2$*, and* $p^m \neq 8$	*2*	*3*
$4p$*; p is an odd prime*	*3*	*4*
pqk; p, q distinct primes, $k \in \mathbb{Z}^+$ *and pqk does not meet any criteria listed above*	*3*	*3*

Proof. Let $n = p$ where p is prime, then $\Gamma(\mathbb{Z}_n) = \emptyset$, since $\mathbb{Z}_p$ is a field.

Let $n = 2^2$, then $\text{diam}(\Gamma(\mathbb{Z}_n)) = 0$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = \infty$ by observation.

Let $n = 3^2$, then $\text{diam}(\Gamma(\mathbb{Z}_n)) = 1$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = \infty$ by observation.

Let $n = p^2$ where p is prime and $p > 3$. Then all zero-divisors of $\mathbb{Z}_{p^2}$ are multiples of p. Consider the zero-divisors m_1p, m_2p, and m_3p. Then, there is a 3-cycle m_1p — m_2p — m_3p — m_1p and it is clear that all zero-divisors are attached to each other, so $\text{diam}(\Gamma(\mathbb{Z}_n)) = 1$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = 3$.

Let $n = 2^3$, then $\text{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = \infty$ by observation.

Let $n = 2p$ where p is an odd prime. Thus, $\mathbb{Z}_n \cong \mathbb{Z}_2 \times \mathbb{Z}_p$. By Theorem 2.5 in [2], we know that $\mathbb{Z}_n$ is a star graph. Thus $\text{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = \infty$.

Let $n = pq$ where p and q are distinct odd primes. Then clearly, $\Gamma(\mathbb{Z}_n)$ is complete bipartite since $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are fields, so $\text{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = 4$.

Let $n = p^m$ where p is prime, $m > 2$, and $p^m \neq 8$. Then, since multiples of p are zero-divisors, consider m_1p and m_2p, any two arbitrary multiples of p. There will always be a 2-path m_1p — p^{m-1} — m_2p and a 3-cycle m_1p^{m-1} — p — m_2p^{m-1} — m_1p^{m-1}. Thus $\text{diam}(\Gamma(\mathbb{Z}_n)) = 2$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = 3$.

Let $n = 4p$. Then $Z(\mathbb{Z}_n) = \{p^\ell, 2p^\ell, 3p^\ell, 2 \cdot 2, 2 \cdot 3, \ldots, 2 \cdot (p-1), 2 \cdot (p+1), \ldots, 2 \cdot (n-1)\}$ for all $\ell \in \mathbb{N}$. Consider 2, $2p^{\ell_1}$, $p^{\ell_2}m_1$, and $2m_2$ where $m_1 \in \{1, 3\}$, m_2 is an even element, and $\ell_i \in \mathbb{N}$. There is a shortest 3-path, namely $p^{\ell_1}m_1$ — $2m_2$ — $2p^{\ell_2}$ — 2, so $\text{diam}(\Gamma(\mathbb{Z}_n)) = 3$. Now consider $2m_3$ where m_3 is a distinct even element such that $m_3 \neq m_2$. There is a 4-cycle $p^{\ell_1}m_1$ — $2m_2$ — $2p^{\ell_2}$ — $2m_3$ — $p^{\ell_1}m_1$. Since the multiples of p must connect to some multiple of 2, and all multiples of 2 must connect to a multiple of p, any cycle in $\Gamma(\mathbb{Z}_n)$ must have an even number of edges. So, since we have a exhibited a 4-cycle and the girth cannot be 3, $\text{g}(\Gamma(\mathbb{Z}_n)) = 4$.

Let $n = pqk$; p, q distinct primes, $k \in \mathbb{Z}^+$ and pqk does not meet any criteria listed above. Then, p — qk — pk — q is a shortest 3-path, and qk — pq — pk is a 3-cycle. Thus $\text{diam}(\Gamma(\mathbb{Z}_n)) = 3$ and $\text{g}(\Gamma(\mathbb{Z}_n)) = 3$. □

5. Conclusions and Acknowledgments

Localizations provide a valuable way to extend rings in general commutative ring theory. Looking at the implications to zero-divisor graphs may provide a deeper understanding of the structure of zero-divisor graphs and, in turn, of the zero-divisors themselves. The classification of modular rings is a step that we hope will lead to a more complete classification of zero-divisor graphs, relying less on number theoric devices and more on broad algebraic properties.

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