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The Nilradical and Non-Nilradical Graphs of Commutative Rings

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Abstract

We introduce two subgraphs of the zero-divisor graph of a commutative ring R: the nilradical and non-nilradical graphs. We examine their connective properties, diameter, and girth in relation to the algebraic properties of R.

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1 Introduction and Definitions

In this paper, R will always denote a commutative ring. An element $z \in R$ is called a *zero-divisor* if there exists a nonzero $r \in R$ such that $rz = 0$. The set of zero-divisors is denoted $Z(R)$ and the set of nonzero zero-divisors as $Z(R)^*$. The *annihilator* of a ring element r is the set of all $s \in R$ such that $rs = 0$. An element is said to be *regular* if it is not a zero-divisor. An element r is *nilpotent* if $r^n = 0$ for some positive integer n. A ring is called *local* if it has a unique maximal ideal. A ring is called *Artinian* if it satisfies the descending chain condition on ideals. Note that all finite rings are Artinian.

The *zero-divisor graph* of R, $\Gamma(R)$, is the graph whose vertices are the nonzero zero-divisors of R, and where two vertices are connected by an edge if and only if their product is 0. Zero-divisor graphs were introduced in [4] by I. Beck, and were redefined to great effect by D.F. Anderson and P. Livingston in [1]. Since then, the concept has been successfully extended to non-commutative rings (e.g., [12]), commutative semigroups (e.g., [6]), ideal-divisor graphs (e.g., [10]), and irreducible divisor graphs (e.g., [5]). Our definition of the zero-divisor graph follows [3] and [9], and differs slightly from that of [1] in that we allow a vertex c to be connected to itself if $c^2 = 0$. Such a vertex is called *looped*.

A *path* between two vertices of a graph G is a sequence of edges that could be followed to get from one vertex to the other. The *distance* between two vertices of a graph G is the number of edges in a minimal path between the vertices. The *diameter* of G (denoted $\operatorname{diam}(G)$) is the maximal distance between any pair of vertices. A *cycle* is a path, excluding loops, from a vertex to itself that does not repeat edges. The *girth* of G (denoted $\operatorname{g}(G)$) is the length of the smallest cycle. If there are no cycles in G, the girth is said to be ∞.

There are some special types of graphs that it will be useful for us to define. A *complete graph* is a graph such that every vertex is adjacent to every other vertex. A *complete bipartite graph* is one where the vertices can be divided into two sets such that every vertex in one set is connected to every vertex in the other, and no vertex is connected to any other in the same set. A *star graph* is a complete bipartite graph in which at least one of the two vertex sets contains only one vertex. That one vertex is called the *center* of the star graph. More information about graph theory may be found in [7].

2 Nilradical and Non-nilradical Graphs

We now introduce two new graphs associated with commutative rings. Both are subgraphs of the zero-divisor graph, and turn out to have surprisingly similar structure to it.

Definition 2.1. *The* nilradical graph*, denoted $N(R)$, is the graph whose vertices are the nonzero nilpotents of R and where two vertices are connected by an edge if and only if their product is 0.*

Definition 2.2. *The* non-nilradical graph*, denoted $\Omega(R)$, is the graph whose vertices are the non-nilpotent zero-divisors of R and where two vertices are connected by an edge if and only if their product is 0.*

Note that $N(R)$ is the subgraph of $\Gamma(R)$ containing only the nilpotent vertices and incident edges and that $\Omega(R)$ is the subgraph of $\Gamma(R)$ containing only the vertices that are not nilpotent and incident edges. In addition, every nonzero zero-divisor of R is a vertex in exactly one of $N(R)$ or $\Omega(R)$.

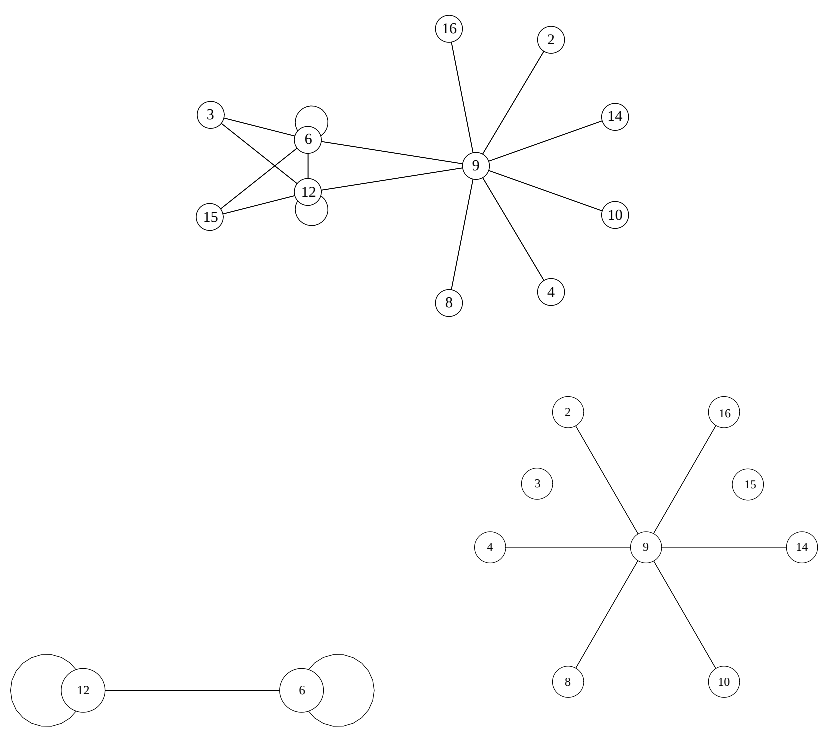

Figure 1: The three graphs of $\mathbb{Z}_{18}$: $\Gamma(\mathbb{Z}_{18})$ (on top), $N(\mathbb{Z}_{18})$ (lower left), and $\Omega(\mathbb{Z}_{18})$ (lower right).

The following theorem gives strong restrictions on the possible diameter and girth of $N(R)$. It provides an alternate proof of Theorems 17 and 21 of [2], which concern domainlike rings, and is a stronger result.

Theorem 2.3. *For any commutative ring R, $N(R)$ is connected with diameter less than or equal to 2, and girth 3 or ∞.*

Proof. By Theorems 2.3 and 2.4 of [1] and Theorem 1.4 of [11], for any commutative ring with unity, R, $\operatorname{diam}(\Gamma(R)) \leq 3$ and $\operatorname{g}(\Gamma(R))$ is 3, 4, or ∞. In fact, the proofs given do not rely on the existence of a multiplicative identity. Since $\operatorname{nil}(R)$ is a finite commutative ring, $N(R)$ is connected with diameter less than or equal to 3 and girth 3, 4, or ∞. If $\operatorname{nil}(R) = \{0\}$, then the theorem holds trivially. Otherwise, by Theorem 2.2 of [3], there exists a nonzero $a \in \operatorname{nil}(R)$ such that $ak = 0$ for all $k \in \operatorname{nil}(R)$. Hence, $\operatorname{diam}(N(R)) \leq 2$.

Suppose $N(R)$ has a cycle of length 4, i.e. $a — b — c — d — a$ for some $a, b, c, d \in \operatorname{nil}(R)\backslash\{0\}$. Then, consider the product bd. If $bd = 0$, then $a — b — d — a$ is a 3-cycle. If $bd = b$, then $bd^m = b$ for all m, contradicting d nilpotent. If $bd = a$, then $a — b — c — a$ is a 3-cycle. Thus, assume without loss of generality that bd is distinct from $0, a, b, c, d$.

Since, bd is nilpotent, $(bd)^n = 0$ for some $n \geq 2$. Suppose $(bd)^2 = 0$. Then, look at b^2d. If $b^2d = 0$, then $a — b — bd — a$ is a 3-cycle. If $b^2d = bd$, then $b^md = bd$ for all m, contradicting b nilpotent. If $b^2d = b$, then $bd = 0$, since $b^2d^2 = 0$, a contradiction. If $b^2d = d$, then $b^{2m}d = d$, contradicting b nilpotent. If b^2d is distinct from $a, b, c, d, bd, 0$, then $a — bd — b^2d — a$ is a 3-cycle. Thus, assume without loss of generality that $b^2d = a$.

Look at b^2. If $b^2 = 0$, then we have a contradiction, since $b^2d = a$. If $b^2 = b$, then b is not nilpotent, a contradiction. If $b^2 = d$, then $d^2 = a$ by substitution, so $a — d — c — a$ is a 3-cycle. If $b^2 = a$, then $ad = a$, contradicting d nilpotent. If $b^2 = c$, then $a — b — c — a$ is a 3-cycle. If $b^2 = bd$, then $a = b^3$, so $a — b — c — a$ is a 3-cycle. If b^2 is distinct from $0, a, b, c, d, bd$, then $a — bd — b^2 — a$ is a 3-cycle. This finishes the case where $(bd)^2 = 0$.

So, assume the least n satisfying $(bd)^n = 0$ is greater than 2. Look at $(bd)^{n-1}$. If $(bd)^{n-1} = 0$, this contradicts our choice of n. If $(bd)^{n-1} = b$, then $a — b — bd — a$ is a 3-cycle. If $(bd)^{n-1} = bd$, then $n = 2$, a contradiction. If $(bd)^{n-1} = a$, then $a — b — c — a$ is a 3-cycle. If $(bd)^{n-1}$ is distinct from a, b, c, d, bd, then $a — bd — (bd)^{n-1} — a$ is a 3-cycle. The remaining cases are analogous to those already considered. □

We now present some definitions that we need to describe the non-nilradical graph, $\Omega(R)$.

Definition 2.4. *A vertex of a graph is* isolated *if it has no edges incident to it.*

Definition 2.5. *A graph is* almost connected *if there exists a path between any two non-isolated vertices.*

Theorem 2.6. *For any a commutative ring R, $\Omega(R)$ is almost connected and the connected component has diameter less than or equal to 3 and girth 3, 4, or ∞.*

Proof. Let $a, b, c, d \in Z(R)\backslash \text{nil}(R)$ be distinct and let $a — b$, $c — d$. We show there is a path of length at most 3 between a and c. Consider bd. Either $bd \in Z(R)\backslash\text{nil}(R)$, or else $bd \in \text{nil}(R)$. If $bd \in Z(R)\backslash\text{nil}(R)$, then $a — bd — c$ is a path of length 2. (If bd is not distinct from either a or c, then we have an even shorter path.) If $bd \in \text{nil}(R)$, then there exists an $n \geq 1$ such that $(bd)^n = 0$. Since b, $d \notin \text{nil}(R)$, b^n and d^n are vertices in $\Omega(R)$. Thus, $a — b^n$ and $c — d^n$, but also $b^n — d^n$. Therefore, $a — b^n — d^n — c$ is a path of length 3. Again if the four elements in the path should fail to be distinct, there exists a shorter path from a to c.

Suppose there exists a cycle, $v_1 — v_2 — \ldots — v_n — v_1$, of length greater than 4 in $\Omega(R)$. Then look at v_2v_n. If v_2v_n is not nilpotent then $v_1 — v_2v_n — v_{n-1} — v_n — v_1$ is a 4-cycle. If v_2v_n happens to be v_1, without loss of generality, $v_1 — v_3 — v_2 — v_1$. If v_2v_n is nilpotent, let k be the least positive integer such that $(v_2v_n)^k = 0$. Then, $v_1 — v_2^k — v_n^k — v_1$ is a 3-cycle, unless, for example, $v_2^k = v_n^k$, in which case, $v_1 — v_2^k — v_3 — v_2 — v_1$. Thus, whenever Ω contains a cycle, there exists a cycle of length no more than 4. □

We refer to the connected portion of $\Omega(R)$ as $\Omega_c(R)$. Note that it satisfies the same diameter and girth conditions as zero-divisor graphs.

3 Categorization of $N(\mathbb{Z}_n)$ by Diameter and Girth

We now focus on modular rings, and classify them by the diameter and girth of their nilradical graphs.

Theorem 3.1. *The following table holds true for $N(\mathbb{Z}_n)$:*

Factorization of n	*Diameter*	*Girth*
$p_1p_2 \ldots p_m$ *such that all* p_i *are distinct primes*	-	-
$4k$, $gcd(2,k) = 1$, $p^2 \nmid k$, *for all prime* p	*0*	∞
$9k$, $gcd(3,k) = 1$, $p^2 \nmid k$, *for all prime* p	*1*	∞
p^2, *p prime*, $p > 3$	*1*	*3*
$2p^2$, *p prime*, $p > 3$	*1*	*3*
p^2q^2, *p and q prime*, $p \neq q$	*1*	*3*
p^2d, $gcd(p,d) = 1$, *p prime*, $p > 3$, *d not divisible by any non-trivial cube*	*1*	*3*
$8k$, $gcd(8,k) = 1$, $p^2 \nmid k$, *for all prime* p	*2*	∞
$p^\ell a$, $\ell \geq 3$, *p prime*, $p > 2$	*2*	*3*
$2^\ell b$, $\ell \geq 3$, *b not a product of distinct primes*	*2*	*3*

Note that because all possible factorizations are considered, these are the only diameter and girth combinations for $N(\mathbb{Z}_n)$, so all combinations are classified.

Proof. Let $n = p_1p_2\dots p_m$ for some positive integer m, such that all p_i are distinct primes. Then, clearly $N(\mathbb{Z}_n)$ is the empty graph.

Let $n = 4k$, $\gcd(2,k) = 1$, $p^2 \nmid k$, for all prime p. Then the nilpotents are the multiples of $2k$. Note that $2(2k) \equiv 0$, $3(2k) \equiv 2k$, and so on. Therefore, the only nilpotents are 0 and $2k$. Thus, $N(\mathbb{Z}_n)$ only has 1 vertex, implying diameter 0, girth ∞.

Let $n = 9k$, $\gcd(3,k) = 1$, $p^2 \nmid k$, for all prime p. Then the nilpotents are the multiples of $3k$. Note that $3(3k) \equiv 0$, $4(3k) = 3(3k) + 3k \equiv 3k$, $5(3k) \equiv 2(3k)$, and so on. Therefore, the only nilpotents are 0, $3k$ and $2(3k)$. Thus, $N(\mathbb{Z}_n)$ only has 2 vertices, implying diameter 1, girth ∞.

Let $n = p^2$, where p is prime and $p > 3$. Then, the nilpotent elements are the multiples of p. Let rp, sp, and tp be distinct in $\mathrm{nil}(\mathbb{Z}_n)$. Consider the 3-cycle, rp — sp — tp — rp. Thus, $\mathrm{g}(N(\mathbb{Z}_n)) = 3$. Since the elements were arbitrary and they are all connected, $\mathrm{diam}(N(\mathbb{Z}_n)) = 1$.

Let $n = 2p^2$, where p is prime and $p > 3$. Then, the nilpotent elements are the multiples of $2p$. Let $2rp$, $2sp$, and $2tp$ be distinct in $\mathrm{nil}(\mathbb{Z}_n)$. Consider the 3-cycle, $2rp$ — $2sp$ — $2tp$ — $2rp$. Thus, $\mathrm{g}(N(\mathbb{Z}_n)) = 3$. Since the elements were arbitrary and they are all connected, $\mathrm{diam}(N(\mathbb{Z}_n)) = 1$.

Let $n = p^2q^2$, where p and q are prime and $p \neq q$. Then, the nilpotent elements are the multiples of pq. Let rpq, spq, and tpq be distinct in $\mathrm{nil}(\mathbb{Z}_n)$. Consider the 3-cycle, rpq — spq — tpq — rpq. Thus, $\mathrm{g}(N(\mathbb{Z}_n)) = 3$. Since the elements were arbitrary and they are all connected, $\mathrm{diam}(N(\mathbb{Z}_n)) = 1$.

Let $n = p^2d$, where p is prime, $p > 3$, $\gcd(p,d) = 1$, and d is not divisible by any non-trivial cube. Then, the nilpotent elements are the multiples of pd. Let rpd, spd, and tpd be distinct in $\mathrm{nil}(\mathbb{Z}_n)$. Consider the 3-cycle, rpd — spd — tpd — rpd. Thus, $\mathrm{g}(N(\mathbb{Z}_n)) = 3$. Since the elements were arbitrary and they are all connected, $\mathrm{diam}(N(\mathbb{Z}_n)) = 1$.

Let $n = 8k$, $\gcd(8,k) = 1$, $p^2 \nmid k$, for all prime p. Then the nilpotents are the multiples of $2k$. Note that $4(2k) \equiv 0$, $5(2k) \equiv 2k$, $6(2k) \equiv 2(2k)$, and so on. Therefore, the only nilpotents are 0, $2k$, $2(2k)$ and $3(2k)$. Thus, $N(\mathbb{Z}_n)$ only has 3 vertices. Notice that $2k$ cannot be connected to $3(2k)$ since the product is not a multiple of 8. Hence $N(\mathbb{Z}_n)$ has diameter 2 and girth ∞.

Let $n = p^\ell a$, where $\ell \geq 3$, p is an odd prime, and $a \in \mathbb{Z}$. Then mp, $2mp$, $mp^{\ell-1}$, and $2mp^{\ell-1}$ are all nilpotent. Since mp is not connected to $2mp$ the diameter is greater than 1. Thus, the diameter is 2 by Theorem 2.3. In addition, mp — $mp^{\ell-1}$ — $2mp^{\ell-1}$ — mp is a 3-cycle. Thus, $\mathrm{g}(N(\mathbb{Z}_n)) = 3$.

Let $n = 2^\ell b$, where $\ell \geq 3$ and $b \in \mathbb{Z}$ not a product of distinct primes. Then $b = cp^2$ for some prime p. Then, $2b$, $6b$, $b2^{\ell-1}$, and $3b2^{\ell-1}$ are all nilpotent. Since $2b$ is not connected to $6b$ the diameter is greater than 1. Thus, the diameter is 2 by Theorem 2.3. In addition, $cp^22^{\ell-1}$ — $cp2^\ell$ — $cp2^{\ell-1}$ — $cp^22^{\ell-1}$ is a 3-cycle. Thus, $\mathrm{g}(N(\mathbb{Z}_n)) = 3$. □

For modular rings where $N(\mathbb{Z}_n)$ has girth ∞, we determine some number theoretic properties of the nilpotents.

Corollary 3.2. *If $N(\mathbb{Z}_n)$ has diameter 0 and girth ∞, then the sole vertex is $\frac{n}{2}$.*

Proof. By Theorem 3.1, we know $n = 4k$, $\gcd(2, k) = 1$, $p^2 \nmid k$, for all prime p. The nonzero nilpotent is $2k$ and $n = 4k$. Thus, the vertex is $\frac{n}{2}$. □

Corollary 3.3. *If $N(\mathbb{Z}_n)$ has diameter 1 and girth ∞, then there are two distinct, nonzero nilpotents and their sum is n.*

Proof. By Theorem 3.1, we know $n = 9k$, $\gcd(3, k) = 1$, $p^2 \nmid k$, for all prime p. The nonzero nilpotent elements are $3k$ and $6k$. Hence, $3k + 6k = 9k = n$. □

Corollary 3.4. *If $N(\mathbb{Z}_n)$ has diameter 2 and girth ∞, then the sum of the two endpoints is n and the center point is $\frac{n}{2}$.*

Proof. By Theorem 3.1, we know $n = 8k$, $\gcd(8, k) = 1$, $p^2 \nmid k$, for all prime p. Then the nilpotent elements are $2k$, $4k$, $6k$, where the center is $4k$. Hence, $2k + 6k = 8k = n$ and $4k = \frac{8k}{2} = \frac{n}{2}$. □

4 Categorization of $\Omega_c(R)$ by Diameter and Girth

We now classify finite rings with unity by the diameter and girth of their non-nilradical graphs. This is a more general classification than was presented for nilradical graphs.

Lemma 4.1. *If R is a finite, local ring with unity, then $\Omega_c(R)$ is empty.*

Proof. By Theorem 2.3 of [3], $Z(R) = \text{nil}(R)$. So, $\Omega_c(R)$ is empty. □

Lemma 4.2. *If R is the direct product of three or more commutative rings with unity, then $\Omega_c(R)$ has diameter 3 and girth 3.*

Proof. Let R be the direct product of $n \geq 3$ such rings. So, $R \cong R_1 \times R_2 \times \cdots \times R_n$. Consider $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, and $(0, 0, 1, 0, \ldots, 0)$, none of which are nilpotent. These form a 3-cycle, so $\text{g}(\Omega_c(R)) = 3$.

Consider $v_1 = (1, 0, 1, 1, \ldots, 1)$ and $v_2 = (1, 1, 0, \ldots, 0)$. Since v_1 is connected to $(0, 1, 0, \ldots 0)$ and v_2 is connected to $(0, 0, 1, 0, \ldots, 0)$, v_1 and v_2 are in $\Omega_c(R)$. Clearly, v_1 and v_2 are not connected. Also, there is no vertex connected to both v_1 and v_2, because the only ring element that annihilates both vertices is $(0, 0, \ldots, 0)$. Therefore, $\text{diam}(\Omega_c(R)) = 3$ □

Theorem 4.3. *For R, a finite commutative ring with unity, $\Omega_c(R)$ is empty, has diameter 1 and girth ∞, diameter 2 and girth ∞, diameter 2 and girth 4, or diameter 3 and girth 3.*

Proof. Since R is finite, R is Artinian and, by Theorem 16.3 of [8], R can be decomposed into a finite direct product of local rings.

Suppose R is itself local. Then $\Omega_c(R)$ is empty by Lemma 4.1.

Suppose $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\Omega_c(R)$ has only two vertices and so has diameter 1 and girth ∞.

Suppose $R \cong \mathbb{Z}_2 \times L$ where L is a local ring not isomorphic to a field. Then the only non-nilpotent zero-divisors are $(1, 0)$, $(1, a)$ where $a \in Z(L)^*$, and $(0, b)$, $b \in L$ such that $b \notin Z(L)$. The only elements of R that annihilate $(1, a)$ are of the form $(0, a')$ where $a' \in Z(L)$ such that $aa' = 0$. But, since L is local, $(0, a')$ is nilpotent, thus $(1, a)$ is an isolated vertex. Then, $(1, 0)$ is attached to all $(0, b)$ vertices, and none of the $(0, b)$ vertices are connected to each other. Thus, $\Omega_c(R)$ is a star graph and has diameter 2 and girth ∞.

Suppose $R \cong F \times L$ where F is a field not isomorphic to $\mathbb{Z}_2$ and L is a local ring that is not a field. Then the only non-nilpotent zero-divisors are $(f, 0)$, (f, a) where $f \in F$, $a \in Z(L)^*$, and $(0, b)$, $b \in L$ such that $b \notin Z(L)$. The only elements of R that annihilate (f, a) are of the form $(0, a')$ where $a' \in Z(L)$ such that $aa' = 0$. But, since L is local, $(0, a')$ is nilpotent, thus (f, a) is an isolated vertex. Then, $(f, 0)$ is attached to all $(0, b)$ vertices, none of the $(0, b)$ vertices are connected to each other, and none of the $(f, 0)$ vertices are connected to one another. Thus, $\Omega_c(R)$ is complete bipartite and has diameter 2 and girth 4.

Suppose $R \cong F_1 \times F_2$ where F_1, F_2 are fields not isomorphic to $\mathbb{Z}_2$. Then none of the nonzero zero-divisors are nilpotent. Thus, $\Gamma(R) \cong \Omega_c(R)$ is complete bipartite and has diameter 2 and girth 4.

Suppose $R \cong L_1 \times L_2$ where L_1, L_2 are local rings, neither of which is a field. The only non-nilpotent zero-divisors are elements of the form $(r_1, 0)$, (r_1, b) (a, r_2), $(0, r_2)$ where r_1, r_2 are regular and $a \in Z(L_1)^*$, $b \in Z(L_2)^*$. The only elements of R that annihilate (r_1, b) are of the form $(0, b')$ where $b' \in Z(L)$ such that $bb' = 0$. But, since L is local, $(0, b')$ is nilpotent, thus (r_1, b) is an isolated vertex. A similar argument shows that all elements (a, r_2) are isolated. Every vertex of the form $(r_1, 0)$ is attached to every vertex of the form $(0, r_2)$ and to no vertex of the form $(r_1, 0)$. Similarly, every vertex of the form $(0, r_2)$ is attached to every vertex of the form $(r_1, 0)$ and to no vertex of the form $(0, r_2)$. Thus, $\Omega_c(R)$ is complete bipartite and has diameter 2 and girth 4.

Suppose $R \cong L_1 \times L_2 \times \ldots \times L_n$ where $n \geq 3$ and L_i are all local rings. Since R has a unity element, there must be a unity element in every L_i. So, by Lemma 4.2, $\Omega_c(R)$ has diameter 3 and girth 3. $\square$

The following table provides a categorization of $\Omega_c(\mathbb{Z}_n)$ by diameter and girth. This classification follows directly from application of Theorem 4.3. Note the restricted diameter, girth combinations as compared to the table of $N(\mathbb{Z}_n)$ provided in Theorem 3.1.

Corollary 4.4. *The following table holds true for $\Omega_c(\mathbb{Z}_n)$:*

Factorization of n	*Diameter*	*Girth*
p^m, *where p is prime and* $m \in (Z)^+$	-	-
$2p^k$, *where p is an odd prime and* $k > 1$	*2*	∞
$p^k q^\ell$, *where p, q are distinct primes,* $k, \ell \in \mathbb{Z}^+$, *and* $p^k, q^\ell \neq 2$	*2*	*4*
$p_1^{e_1} p_2^{e_2} \dots p_c^{e_c}$, *all* p_i *are distinct,* $e_i \in \mathbb{Z}^+$, *and* $c \geq 3$	*3*	*3*

Corollary 4.5. *If $\Omega_c(\mathbb{Z}_n)$ has diameter 2 and girth ∞, then the center of the star graph is $\frac{n}{2}$.*

Proof. Since $\Omega_c(\mathbb{Z}_n)$ has diameter 2 and girth ∞, it is clearly a star graph. By the above table, $n = 2p^k$, where p is prime and $k > 1$. So, $\frac{n}{2}$ is connected to every multiple of 2, and since $\Omega_c(\mathbb{Z}_n)$ is a star graph, anything connected to more than one other vertex must be the center. □

Corollary 4.6. *If $\Omega_c(\mathbb{Z}_n)$ has diameter 2 and girth ∞, then either $n = 18$ or $N(\mathbb{Z}_n)$ has diameter 2 and girth 3.*

Proof. This follows from a comparison of the tables of Theorem 3.1 and Corollary 4.4. □

The following example shows that the restrictions placed on diameter and girth combinations in finite rings need not extend to infinite rings. We present an infinite ring R where $\Omega_c(R)$ has diameter 3 and girth 4. It has not yet been determined whether there exists a ring S such that $\Omega_c(S)$ has diameter 1 and girth 3, diameter 2 and girth 3, or diameter 3 and girth ∞, but it seems that any such ring would have to be highly artificial. Further work on classifying the diameter and girth of $\Omega_c(R)$ for infinite rings would undoubtably be valuable.

Example 4.7. *Let $R = \mathbb{Z}_2[x, y, z]/(xy, yz, xz, x^3)$. Then, diam$(\Omega_c(R)) = 3$ and g$(\Omega_c(R)) = 4$. The zero-divisors of R are the set $\{a_1 x + a_2 x^2 + \sum_{i=1}^{m} b_i y^i + \sum_{i=1}^{n} c_i z^i \mid a_i, b_i, c_i \in \mathbb{Z}_2\}$. The nonzero nilpotents are x, x^2, and $x + x^2$.*

Note that the isolated vertices of $\Omega(R)$ are the zero-divisors containing nonzero powers of both y and z in the sum because they are only connected to nilpotents. One example of a 4-cycle in $\Omega_c(R)$ is y — z — $x + y$ — $x^2 + z$ — y. Any polynomial containing y can only be connected to a nilpotent element or a polynomial containing z. Similarly, any polynomial containing z can only be connected to a nilpotent element or a polynomial containing y. Therefore, every cycle must have an even number of edges, and hence g$(\Omega_c(R)) = 4$.

There is a 3-path between $x + x^2 + z$ and $x + y$ in $\Omega_c(R)$, namely $x + x^2 + z$ — y — z — $x + y$. Since there cannot be a vertex in $\Omega_c(R)$ that is connected to both $x + x^2 + z$ and $x + y$, there is not a shorter path between them. So, diam$(\Omega_c(R)) = 3$.

5 Further Properties of $\Omega(R)$

The previous section classified $\Omega_c(R)$ according to its diameter and girth. This section considers other interesting properties of the non-nilradical graph for finite rings with unity.

Theorem 5.1. *For R, a finite commutative ring with unity, $\Omega(R)$ contains at least one isolated vertex if and only if $nil(R) \neq \{0\}$ and $nil(R) \neq Z(R)$, that is to say, if and only if $\Omega(R)$ is a proper subgraph of $\Gamma(R)$.*

Proof. Since R is finite, R is Artinian and, by Theorem 16.3 of [8], can be decomposed into a finite direct product of local rings.

Suppose R is itself local. Then $\Omega(R)$ is empty by Theorem 4.1 and there are no isolated points.

Suppose $R \cong \mathbb{Z}_2 \times L$ where L is a local ring not isomorphic to a field. Then the only non-nilpotent zero-divisors are $(1,0)$, $(1,a)$ where $a \in Z(L)^*$, and $(0,b)$, $b \in L$ such that $b \notin Z(L)$. The only elements of R that annihilate $(1,a)$ are of the form $(0,a')$ where $a' \in Z(L)$ such that $aa' = 0$. But, since L is local, $(0,a')$ is nilpotent; thus $(1,a)$ is an isolated vertex.

Suppose $R \cong F \times L$ where F is a field not isomorphic to $\mathbb{Z}_2$ and L is a local ring that is not a field. The only non-nilpotent zero-divisors are $(f,0)$, (f,a) where $f \in F$, $a \in Z(L)^*$, and $(0,b)$, $b \in L$ such that $b \notin Z(L)$. The only elements of R that annihilate (f,a) are of the form $(0,a')$ where $a' \in Z(L)$ such that $aa' = 0$. But, since L is local, $(0,a')$ is nilpotent; thus (f,a) is an isolated vertex.

Suppose $R \cong L_1 \times L_2$ where L_1, L_2 are local rings, neither of which is a field. The only non-nilpotent zero-divisors are elements of the form $(r_1,0)$, (r_1,b) (a,r_2), $(0,r_2)$ where r_1, r_2 are regular and $a \in Z(L_1)^*$, $b \in Z(L_2)^*$. The only elements of R that annihilate (r_1,b) are of the form $(0,b')$ where $b' \in Z(L)$ such that $bb' = 0$. But, since L is local, $(0,b')$ is nilpotent; thus (r_1,b) is an isolated vertex.

Suppose $R \cong L_1 \times L_2 \times \ldots \times L_n$ where $n \geq 3$, L_i are all local rings, and at least one L_i is not a field. Since R has a unity element, there must be a unity element in every L_i. Assume, without loss of generality, L_1 is not a field. Then L_1 has a nonzero zero-divisor a. So, $(a,1,1,\ldots,1)$ is an isolated vertex as it is only annihilated by elements of the form $(a',0,0,\ldots,0)$ where $aa' = 0$, all of which are nilpotent.

Suppose $R \cong F_1 \times F_2 \times \ldots \times F_n$ where $n \geq 1$ and F_i are all fields. None of the nonzero zero-divisors are nilpotent. So, $\Gamma(R) \cong \Omega(R)$ has no isolated vertices.

Therefore, isolated vertices occur exactly when R is of the form $\mathbb{Z}_2 \times L$, $F \times L$, $L_1 \times L_2$, or $L_1 \times L_2 \times \ldots \times L_n$. These are exactly the forms for which $\text{nil}(R) \neq \{0\}$ and $\text{nil}(R) \neq Z(R)$. □

In [4], where Beck introduced zero-divisor graphs, he focused on colorings and the chromatic number of the graphs. However, strong results about the chromatic number of $\Gamma(R)$ have not been forthcoming. Here we find, however, that $\Omega(R)$ is mysteriously more amenable to the study of chromatic number.

Theorem 5.2. *Let R be an non-local finite commutative ring with unity. Then, $\chi(\Omega(R)) = n$, where n is the number of local rings in the decomposition of R.*

Proof. Decompose R into a direct product of local rings, $L_1, L_2, \ldots, L_n$, where $n \geq 2$. Since R has a unity element, every L_i does as well. Then $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, and $(0, 0, \ldots, 0, 1)$ form an n-clique. So, $\chi(\Omega(R)) \geq n$.

We show that, in fact, $\chi(\Omega(R)) = n$ by presenting a n-coloring. Name the n colors c_1, c_2, $\ldots$, c_n. Consider any vertex of $\Omega(R)$, $(a_1, a_2, \ldots, a_n)$. Color this vertex with the color c_i, where i is the least integer such that a_i is not nilpotent. Note that there always is such an i, since otherwise all the components would be nilpotent and the vertex would not be in $\Omega(R)$. To show this is an n-coloring, it suffices to show that any two vertices with the same coloring are not adjacent. Assume $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are adjacent with the same coloring c_k. Then, $(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = 0$. Therefore, $a_k \cdot b_k = 0$. However, this is not possible, as a_k and b_k are not nilpotent by the definition of our coloring, and so are not zero-divisors, since L_k is a finite local ring. □

Theorem 5.3. *Let R be a finite commutative ring with unity, $\Omega_c(R) \cup \{0\}$ is multiplicatively closed, unless R is isomorphic to the direct product of three or more local rings, $L_1, L_2, L_3, \ldots L_n$, and $\Gamma(L_i)$ is not complete for some i.*

Proof. Since R is finite, R is Artinian and, by Theorem 16.3 of [8], R can be decomposed into a finite direct product of local rings.

Suppose R is itself local. Then $\Omega(R)$ is empty by Theorem 4.1 and thus closed under multiplication.

Suppose $R \cong F \times L$, where F is a field and L is a local ring that is not a field. Then the only non-nilpotent zero-divisors are $(f, 0)$, (f, a) where $f \in F$, $a \in Z(L)^*$, and $(0, b)$, $b \in L$ such that $b \notin Z(L)$. The only elements of R that annihilate (f, a) are of the form $(0, a')$, where $a' \in Z(L)$ such that $aa' = 0$. But, since L is local, $(0, a')$ is nilpotent; thus (f, a) is an isolated vertex. So, $\Omega_c(R)$ is made up of elements of the form $(f, 0)$ and elements of the form $(0, b)$ where b is not nilpotent. Clearly, $(f_1, 0) \cdot (f_2, 0)$ is a vertex of $\Omega_c(R)$ for all f_i. In addition, $(f, 0) \cdot (0, b) = (0, 0)$ for all b, f. It remains to check closure for products of the form $(0, b) \cdot (0, b')$ where b and b' are not nilpotents. Assume $(0, b) \cdot (0, b')$ is nilpotent. Then $(bb')^n = 0$ for some n. So, $b \cdot (b^{n-1} b'^n) = 0$. Hence, b is a zero-divisor. However, since L is a finite local ring by Theorem

2.3 [3], b is nilpotent, a contradiction. Thus, $\Omega_c(R) \cup \{0\}$ is multiplicatively closed.

Suppose $R \cong L_1 \times L_2$, where L_1, L_2 are local rings, neither of which is a field. The only non-nilpotent zero-divisors are elements of the form $(r_1, 0)$, (r_1, b) (a, r_2), $(0, r_2)$ where r_1, r_2 are regular and $a \in Z(L_1)^*$, $b \in Z(L_2)^*$. The only elements of R that annihilate (r_1, b) are of the form $(0, b')$, where $b' \in Z(L)$ such that $bb' = 0$. But, since L is local, $(0, b')$ is nilpotent; thus (r_1, b) is an isolated vertex. So, $\Omega_c(R)$ is made up of elements of the form $(r_1, 0)$ and $(0, r_2)$. In addition, $(r_1, 0) \cdot (0, r_2) = (0, 0)$ for all r_1 and r_2. It remains, without loss of generality, to check closure for products of the form $(r_1, 0) \cdot (r_1', 0)$. Assume $(r_1, 0) \cdot (r_1', 0)$ is nilpotent. Then $(r_1 r_1')^n = 0$ for some n. So, $r_1 \cdot (r_1^{n-1} {r_1'}^n) = 0$. Hence, r_1 is a zero-divisor, a contradiction. Thus, $\Omega_c(R) \cup \{0\}$ is multiplicatively closed.

Suppose $R \cong L_1 \times L_2 \times \ldots \times L_n$, where $n \geq 3$, L_i are all local rings, and at least one L_i is not a field. In addition, suppose $\Gamma(L_i)$ is complete for all i. Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be any two elements in R and suppose $(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$ is nilpotent. Then $a_i b_i$ is nilpotent for every i. For every i, either at least one of a_i, b_i is zero or else a_i and b_i are both nonzero nilpotents. For all j, if a_j is regular, then b_j must be 0. Therefore, $a_j b_j = 0$. If a_j is not regular, then it is a zero-divisor and since $\Gamma(L_j)$ is complete, again $a_j b_j = 0$. Hence, $a_i b_i = 0$ for all i. Thus, if the product of any two vertices of $\Omega_c(R)$ is nilpotent,it must be 0. Therefore, $\Omega_c(R) \cup \{0\}$ is multiplicatively closed.

Suppose $R \cong F_1 \times F_2 \times \ldots \times F_n$, where $n \geq 1$ and F_i are all fields. None of the nonzero zero-divisors are nilpotent. So, trivially, $\Omega_c(R) \cup \{0\}$ is multiplicatively closed. □

We have exhibited two very well-behaved subgraphs of the zero-divisor graph, $N(R)$ and $\Omega(R)$. There is no *a priori* reason to expect these graphs to have meaningful graphical structure. In particular, the vertices of $\Omega(R)$ are chosen by taking the weakly structured set $Z(R)$ and removing an ideal; even adjoining 0, this set of vertices has almost no algebraic structure, being closed under neither addition nor multiplication. However, both graphs (particularly $\Omega(R)$) are highly structured. The ultimate reason for this structure is unclear and should be investigated. A truly satisfying explanation of why there is so much structure where it does not seem there ought to be any, would likely give great insight into the set $Z(R)$.

For example, in [9], B. Kelly and E. Wilson conjectured that finite isomorphic zero-divisor graphs must result from multiplicatively isomorphic sets of zero-divisors. Determining a graphical algorithm whereby one could dissect a finite zero-divisor graph into nilradical and non-nilradical subgraphs would be great progress towards a proof of this conjecture, as the conjecture implies that the two subgraphs are uniquely determined by the zero-divisor graph. After breaking down the zero-divisor graph into such nicely structured subgraphs, the result would only need to be shown for graphs smaller than was begun with, and perhaps some sort of induction argument on the cardinality of the graph would apply.

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