Five Problems on a 2-Integrator Differential Analyzer and the Sign of Direction

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February 16, 2007

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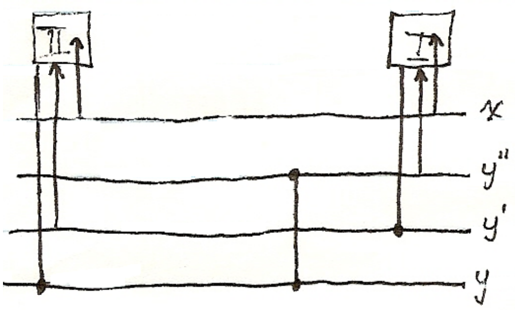
1 The Default Setup and Program I



The default setup of the differential analyzer is the setup for the circle test, and it will be used as the baseline for all changes for the other programs. It is set up so that the output of each integrator proportionally affects the input of the other integrator, and the inputs vary the output each respective integrator continuously. This "back and forth" transfer of motion (through shaft rotations) demonstrates the essence of a particular differential equation: $\frac{d^2y}{dx^2} = -y$

because the rotations of the shaft representing the second derivative are the same rotations of the shaft representing y. The solution of this equation yields a linear combination $y = Asin(x) + Bcos(x)$ is a trigonometric function that behaves normally. If one were to graph y versus x, the solution would be a wave of sine and cosine, but typically, the graph for this particular equation is made from graphing y versus $\frac{dy}{dx}$. This yields a circle of the form

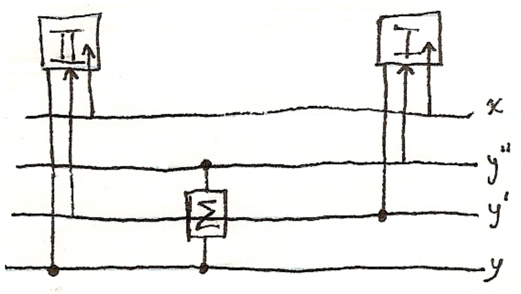
$$\begin{cases} x = \quad Acos(t) + Bsin(t) \\ y = -(Acos(t) + Bsin(t)) \end{cases}$$

that is used to test the accuracy of the machine by determining how close to completing the circle perfectly when the circle reaches its starting point.

2 Independent Variable Setup

To graph any solution versus the independent variable, one must gear the machine to plot the independent varible directly to the plotter. This is done by engaging the worm gear that is hanging on the rod that carries the indepdendent variable's motion from the motor to the gear below it. Then, by disengaging the small gear on the y-rod that carries the motion from the second integrator across the machine from the gear that sits below it. This change removes the output to the plotter from the rod carrying the shaft rotations representing the first derivative to the continuous rotation of the independent variable. Doing this keeps our plotted graphs from being parametric, and draws a graph of the independent variable versus the dependent variable – a direct graph of the solution!

3 Program II - $\frac{d^2y}{dx^2} = -y - k \cdot \frac{dy}{dx}$, k is a constant



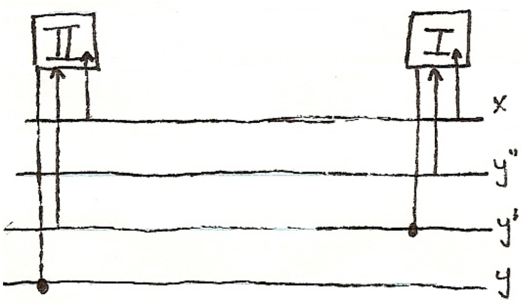
3.1 The Sum

This differential equation involves a sum, and requires the use of the differential settled in the middle of the machine. A sum is represented in the machine through the differential in the center by transferring the motion from integrator I (y') to the differential along with the motion from integrator II (y) that already affects the differential. When both sets of motion are sent to the differential, the two sides of it spin indepdendently, and the sum of the rotations spin the center of the differential – the only piece attached to the rod.

3.2 The Gearing

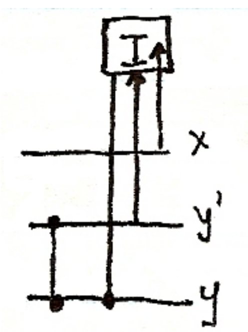
To set this problem up, the independent variable must be tapped to the plotter as described in section 3. Next, disengage the "locking gear" hanging from the machine on the power switch side of the differential analyzer so that the gear it is meshed with is allowed to turn so that the motion that becomes unlocked can carry shaft rotations to the integrator I side of the differential, allowing a sum to occur. The rotations of the rod the sum occurs on is sent to integrator I because of the equation, and completes the system.

4 Program III - $\frac{d^2y}{dx^2} = c$, c is a constant



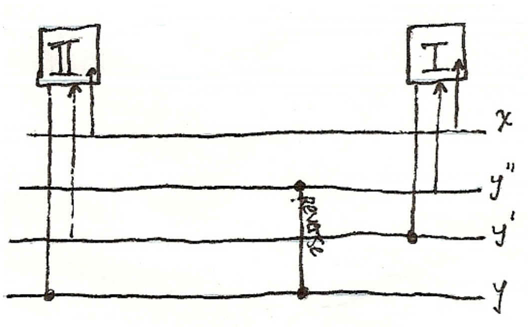
In order to graph a parabola $y = ax^2 + bx + c$, gearing the machine to solve $\frac{d^2y}{dx^2} = c$, c is a constant must be considered. Initially, to gear the machine for this setup, the independent variable setup must be done. After that, the only change that needs to be done is to disengage the small gears that turn the lead screw to integrator II. This change eliminates any variation of the input to integrator II, and since the default setup of the machine is symmetric, we can call integrator II y'' and since the input does not vary, y'' is constant. The rest of the mechanics follow the same as the default setup – the output of integrator II affects the input of integrator I.

5 Program IV - $\frac{dy}{dx} = ky$, k is a constant



In order to graph an exponential, e^{kx}, the solution of $\frac{dy}{dx} = ky$ must be considered. First, set up independent variable graphing so the solution can be graphed versus the independent variable. Since this equation is only of first degree, only one integration is necessary to achieve the solution. To stop either integrator for operating, simple disengage the gear meshed to the worm gear attached to the rod holding the integrator plate that rotates parallel to the table to stop the integrator from integrating, and disengage the small gears that turn the lead screw to integrator II to stop input from possibly physically limiting the plot of the solution. With one integrator stopped, the other integrates y' to y yielding an exponential.

6 Program V - $\frac{d^2y}{dx^2} = y$



To graph $\frac{d^2y}{dx^2} = y$, the only changes to the default setting that need to be made are setting up independent variable graphing and reversing the direction of the lead screw of one of the integrators by attaching an odd number of 1:1 gears to the input gearing of the lead screw rod which will reverse the motion and change program I into program V because the sign on the dependent variable represents shaft rotations, and reversing a shaft rotation will reverse the sign.

7 Positive and Negative Directions

The sign of each initial position is decided by whether the initial position is set to be a distance x from the center of the integrator disk toward either the center of the machine or the outsides of the machine. It is arbitrary as to which directions are positive or negative, but consistency must be maintained. Assume initial position away from the center is positive and initial position toward the center is negative, and the machine is geared to solve $\frac{d^2y}{dx^2} = c$, c is a constant. Setting integrator I (the one that varies with this setup) positive (away from the center of the machine) and integrator II (the constant one) also positive yields a graph of only the positive side of a parabola. Setting both initial position disks negative will yield a similar graph, but arbitrarily will be the same parabola for when $x < 0$. Setting one integrator positive and the other negative (toward the center) will yield a graph of the actual critical point of the parabola, showing the change in the derivative from positive to negative or negative to positive. This can be described with simple calculus: assume integrator I input and integrator II output $= y'$, integrator I ouput $= y$, and integrator II input $= y''$, a constant. y'' represents the concavity of the parabola, y' the slope, and y the parabola itself.
To examine why only half of a parabola is graphed when both inputs are positive, consider where on the parabola $y = ax^2 + bx + c, a > 0$ the conditions $y' =$

$2ax + b > 0$ and $y'' = 2a > 0$ are met. It is then clear to see why the graph is only partial – there are no negatives considered in y's derivatives! It starts positive, increasing, concave up, and continues in that manner until the machine is turned off.

If the initial condition of integrator I is positive and integrator II negative, then the slope of y is initially positive, but the concavity is negative. Since $y' = 2ax + bx > 0$, $y'' = 2a$ and 2a ¡ 0 (by the setting of the initial condition), there will be a critical point at $x = 0$ where the minima (if the coefficent of x^2 is positive) or maxima (if the coefficent of x^2 is negative) of the function is reached.