SOLUTION TO CONICAL TANK DRAINING INTO CYLINDRICAL TANK RELATED RATE PROBLEM

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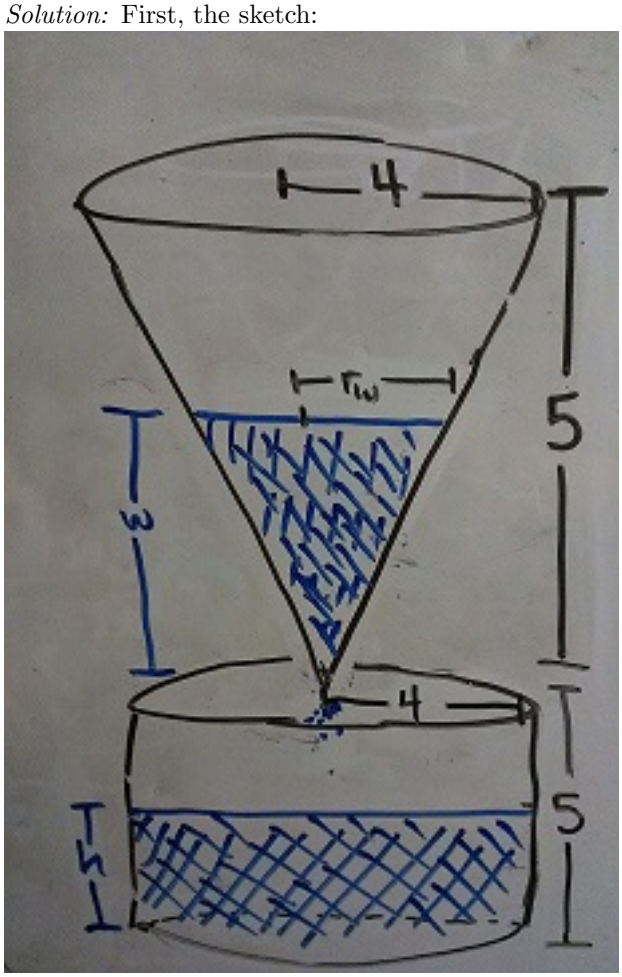
Problem: A concial tank with an upper radius of 4m and a height of 5m drains into a cylindrical tank with a radius of 4m and a height of 5m. If the water level in the conical tank drops at a rate of $0.5\frac{\text{m}}{\text{min}}$, at what rate does the water level in the cylinrical tank rise when the water level in the conical tank is 3m?

Solution: First, the sketch:



The variables in this problem:

- w height of the water in the conical tank
- r_w radius of the water in conical tank when the water there is at height w
- h height of the water in the cylindrical tank.

Note: we don't need a variable for radius of water in cylindrical tank because it's always 4 and hence does not change with time.

The given rate in this problem: $\dfrac{dw}{dt} = -0.5$

We are asked to find: $\dfrac{dh}{dt}$ when $w = 3$

Relevant equations:

- Volume of a cone of radius r_w and height w: $V_{cone} = \dfrac{1}{3}\pi r_w^2 w$
- Volume of a cylinder of radius r and height h: $V_{cyl} = \pi r^2 h$
- Relationship between w and r_w: similar triangles in the diagram yield
$$\frac{r_w}{w} = \frac{4}{5},$$
or equivalently,
$$r_w = \frac{4}{5}w$$
(*note: we need this because we were only given information for* $\dfrac{dw}{dt}$ *not also* $\dfrac{dr_w}{dt}$)
- Relationship between $\dfrac{dV_{cone}}{dt}$ and $\dfrac{dV_{cyl}}{dt}$: first realize that $\dfrac{dV_{cone}}{dt}$ will be negative since it is leaking water and $\dfrac{dV_{cyl}}{dt}$ will be positive since it is collecting the leaking water. Now realize that all water leaked from the conical tank lands in the cylindrical tank, and so all volume that leaves the top ends up in the bottom. Therefore the rate of leaking from the conical tank must equal the rate of water volume accumulating in the cylindrical tank. So we must have the relationship
$$\frac{dV_{cyl}}{dt} = -\frac{dV_{cone}}{dt}.$$
We include the negative sign since $\dfrac{dV_{cone}}{dt}$ is negative and we want $\dfrac{dV_{cyl}}{dt}$ to be positive.

Finding the answer: Using the relationship between w and r_w to eliminate the variable r_w from the equation defining V_{cone}, we see
$$V_{cone} = \frac{1}{3}\pi r_w^2 w = \frac{16}{75}\pi w^3,$$
and now differentiation with respect to t and noting that $\dfrac{dw}{dt} = -0.5$ yields
$$(*) \qquad \frac{dV_{cone}}{dt} = \frac{16}{25}\pi w^2 \frac{dw}{dt} = -\frac{8}{25}\pi w^2.$$
We now consider the equation for V_{cyl} with radius $r = 4$ and height h and see
$$V_{cyl} = 16\pi h$$
and so differentiation with respect to t yields
$$\frac{dV_{cyl}}{dt} = 16\pi \frac{dh}{dt},$$

or in other words, since we were asked to find $\frac{dh}{dt}$, we write

$$(**) \qquad \frac{dh}{dt} = \frac{1}{16\pi}\frac{dV_{cyl}}{dt}.$$

By the relationship between $\frac{dV_{cone}}{dt}$ and $\frac{dV_{cyl}}{dt}$ we see from equation $(*)$ that

$$\frac{dV_{cyl}}{dt} = -\frac{dV_{cone}}{dt} = \frac{8}{25}\pi w^2.$$

Therefore in equation $(**)$ we see

$$\frac{dh}{dt} = \frac{1}{16\pi}\frac{8}{25}\pi w^2 = \frac{1}{50}w^2.$$

Thus we can now easily see the solution: we were asked to find $\frac{dh}{dt}$ when $w = 3$, so we see

$$\left.\frac{dh}{dt}\right|_{w=3} = \frac{1}{50}3^2 = \frac{9}{50}.$$

Answer question in form of a sentence: The water level in the cylindrical tank rises at a rate of $\frac{9}{50}\frac{\text{m}}{\text{min}} = 0.18\frac{\text{m}}{\text{min}}$.