HW9 MTH452/552

Friday, April 19, 2024 3:02 PM

p.291 #1 $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$

$\alpha = \sqrt{2} \rightarrow \alpha^2 = 2 \rightarrow \alpha$ root of $x^2 - 2$

$\rightarrow \{1, \sqrt{2}\}$ basis of $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$

$\rightarrow [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$

#2 $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$

$$\mathbb{Q}(\sqrt{2},\sqrt{3}) = (\mathbb{Q}(\sqrt{2}))(\sqrt{3}) = \{a + \sqrt{2}b + \sqrt{3}c : a,b,c \in \mathbb{Q}\}$$

By similar argument from earlier

$\mathbb{Q}(\sqrt{2})$ has basis $\left[\begin{array}{l}\{1,\sqrt{2}\} \\ [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2\end{array}\right.$

$(\mathbb{Q}(\sqrt{2}))(\sqrt{3})$ has basis $\left[\begin{array}{l}\{1,\sqrt{3}\} \\ [\underbrace{(\mathbb{Q}(\sqrt{2}))(\sqrt{3})}_{=\mathbb{Q}(\sqrt{2},\sqrt{3})} : \mathbb{Q}(\sqrt{2})] = 2\end{array}\right.$

Therefore by Thm 31.4 with $\begin{cases} K = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \\ F = \mathbb{Q} \\ E = \mathbb{Q}(\sqrt{2}) \end{cases}$, we have

degree: $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$

$= 2(2) = 4$

basis: for basis: Thm 31.4 $\Rightarrow$ basis for $\mathbb{Q}(\sqrt{2},\sqrt{3})$ is given by multiplying the bases we have above together, i.e.,

$\{1,\sqrt{2}\}$ and $\{1,\sqrt{3}\}$

$\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ ← size 4, as expected!

#3 $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{18})$ over $\mathbb{Q}$

Cor 31.6 $\Rightarrow [\mathbb{Q}(\sqrt{2},\sqrt{3},\sqrt{18}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{18},\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2},\sqrt{3})][\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$

basis $\{1\}$ — basis $\{1,\sqrt{3}\}$ — basis $\{1,\sqrt{2}\}$

not $\{1, \sqrt{18}\}$ b/c $\sqrt{18} = 3\sqrt{2}$ so we get "nothing new" by tossing in $\sqrt{18}$

$= (1)(2)(2) = 4$

Thm 31.4 $\Rightarrow$ form basis for $\mathbb{Q}(\sqrt{18}, \sqrt{3}, \sqrt{2})$ as vector space over $\mathbb{Q}$ by multiplying everything together:

$\{1, \sqrt{3}, \sqrt{2}, \sqrt{6}\}$ ← size 4, as expected

#4 $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$ over $\mathbb{Q}$

$[\mathbb{Q}(\sqrt[3]{2},\sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{2},\sqrt{2}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$

basis $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$ — 3 elements b/c $\mathrm{irr}(\sqrt[3]{2}, \mathbb{Q}) = x^3 - 2$

basis $\{1, \sqrt{2}\}$ — 2 elts b/c $\mathrm{irr}(\sqrt{2}, \mathbb{Q}) = x^2 - 2$

$= (3)(2) = 6$

Thm 31.4 $\Rightarrow$ form basis for $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$ over $\mathbb{Q}$ is

$\{1, \sqrt{2}, \sqrt[3]{2}, (\sqrt[3]{2})^2,$

$\{1, 2^{1/2}, 2^{1/3}, 2^{2/3}, 2^{5/6}, 2^{1/6}\}$

$= \{1, 2^{1/6}, 2^{2/6}, 2^{3/6}, 2^{4/6}, 2^{5/6}\}$ ← size 6 as expected!

$2^{1/2}, 2^{1/3}, 2^{2/3}$

$(2^{1/2})(2^{1/3}) = 2^{1/2 + 1/3}$

$= 2^{5/6}$

$2^{5/6} = 2^{1/6}2^{4/6}$

$= 2^{1/6}2^{2/3}$

so

$2^{1/6} = (2^{5/6})(2^{2/3})^{-1} \in \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$

#5 similar to #4

#6 $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}$

$\alpha = \sqrt{2} + \sqrt{3}$

$\alpha^2 = 2 + 2\sqrt{6} + 3$

$\alpha^2 - 5 = 2\sqrt{6}$

$\alpha^4 - 10\alpha + 25 = 24$

$\rightarrow \alpha$ root of $x^4 - 10x + 1 \in \mathbb{Q}[x]$

$\Rightarrow [\mathbb{Q}(\sqrt{2}+\sqrt{3}, \mathbb{Q}) : \mathbb{Q}] = 4$

$\Rightarrow$ basis $\{1, \alpha, \alpha^2, \alpha^3\} = \{1, \sqrt{2}+\sqrt{3}, (\sqrt{2}+\sqrt{3})^2, (\sqrt{2}+\sqrt{3})^3\}$

#11 $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ over $\mathbb{Q}(\sqrt{3})$

$\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \{a + (\sqrt{2}+\sqrt{3})b : a,b \in \mathbb{Q}\}$ $\quad$ $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a,b \in \mathbb{Q}\}$

Claim: $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$

Clearly $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$.

Proof: Since $(\sqrt{2}+\sqrt{3})^2 = 5 + 2\sqrt{6} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$, we conclude that $\sqrt{6} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

Now $\sqrt{6}(\sqrt{2}+\sqrt{3}) - 2(\sqrt{2}+\sqrt{3}) = 2\sqrt{3} + 3\sqrt{2} - 2\sqrt{2} - 2\sqrt{3} = \sqrt{3}$,

so $\sqrt{3} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

Also $(\sqrt{2}+\sqrt{3}) - \sqrt{3} = \sqrt{2} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

So $\sqrt{2}, \sqrt{3} \in \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

This means $\mathbb{Q}(\sqrt{2},\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}+\sqrt{3})$.

This completes the proof of the claim.

So we are really being asked about

$(\mathbb{Q}(\sqrt{2}))(\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$ over $\mathbb{Q}(\sqrt{3})$

$[\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{3})] = 2$

basis $\{1, \sqrt{2}\}$

#12 Since $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$,

$[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}(\sqrt{2},\sqrt{3})] = 1$

basis $\{1\}$