HW8 MTH452/552

Friday, April 19, 2024 1:56 PM

$\{a+\sqrt{2}b : a,b\in\mathbb{Q}\}$

p.280 | #4 | basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$

$\alpha=\sqrt{2} \rightarrow \alpha^2=2 \rightarrow \alpha$ root of $x^2-2\in\mathbb{Q}[x]$

$\Rightarrow$ by Thm 30.23 (w/ $n=2$), $\{1,\sqrt{2}\}$ is a basis for $\mathbb{Q}(\sqrt{2})$ as a vector space over $\mathbb{Q}$

#5 | $\mathbb{R}(\sqrt{2})$ over $\mathbb{R}$

$\alpha=\sqrt{2} \rightarrow \alpha$ root of $x^2-2\in\mathbb{R}[x]$

$\Rightarrow$ (Thm 30.23) $\{1,\alpha\}$ basis for $\mathbb{R}(\sqrt{2})$ as vector space over $\mathbb{R}$

#6 | $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$

7/10

$\alpha=\sqrt[3]{2} \rightarrow \alpha$ root of $x^3-2\in\mathbb{Q}[x]$

$\Rightarrow$ Thm 30.23 ($n=3$) $\Rightarrow$ basis $\{1,\alpha,\alpha^2\}=\{1,\sqrt[3]{2},(\sqrt[3]{2})^2\}$

#7 | $\mathbb{C}$ over $\mathbb{R}$

$\mathbb{C}=\{a+bi : a,b\in\mathbb{R}\}=\mathbb{R}(i)$

$\alpha=i \rightarrow \alpha^2=-1 \rightarrow \alpha$ root of $x^2+1\in\mathbb{R}[x]$

$\Rightarrow \{1,i\}$ basis for $\mathbb{C}=\mathbb{R}(i)$ as vec space over $\mathbb{R}$

#8 | $\mathbb{Q}(i)$ over $\mathbb{Q}$

$\alpha=i \rightarrow \alpha^2=-1 \Rightarrow \alpha$ root of $x^2+1\in\mathbb{Q}[x]$

$\Rightarrow \{1,i\}$

#9 | $\mathbb{Q}(\sqrt[4]{2})$ over $\mathbb{Q}$

$\alpha=\sqrt[4]{2} \rightarrow \alpha^4=2 \rightarrow \alpha$ root of $x^4-2\in\mathbb{Q}[x]$

$\xrightarrow[(n=4)]{\text{Thm 30.23}} \{1,\alpha,\alpha^2,\alpha^3\}=\{1,\sqrt[4]{2},(\sqrt[4]{2})^2,(\sqrt[4]{2})^3\}$ is basis for $\mathbb{Q}(\sqrt[4]{2})$ as vector space over $\mathbb{Q}$

#10 |

Consider from example 29.19: $p(x)=x^2+x+1\in\mathbb{Z}_2[x]$

p irreducible b/c it has no roots in $\mathbb{Z}_2$

Consider extension field $\{n+\alpha m : n,m\in\mathbb{Z}_2\}=\mathbb{Z}_2(\alpha)$ where α root of x^2+x+1

$\Rightarrow \alpha^2+\alpha+1=0$
$\alpha^2=-\alpha-1$

Turns out $\mathbb{Z}_2(\alpha)$ has 4 elements

$\mathbb{Z}_2(\alpha)=\{0,1,\alpha,1+\alpha\}$

Goal: Find $\mathrm{irr}(\alpha,\mathbb{Z}_2)$

We know α root of x^2+x+1, so $\alpha^2+\alpha+1=0$

$\boxed{\alpha^2=-\alpha-1}$

Now let

$\beta=1+\alpha$

$\beta^2=(1+\alpha)^2=1+2\alpha+\alpha^2$

$=1+2\alpha+(-\alpha-1)$

$=\cancel{1}+2\alpha-\alpha-\cancel{1}$

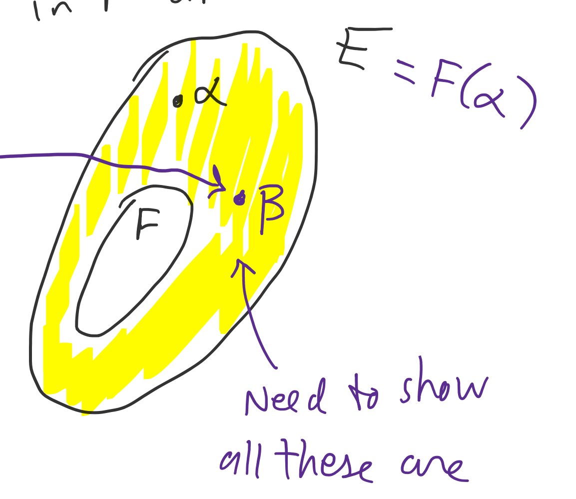
$=\alpha$

Since $\alpha^2+\alpha+1=0$ and $\alpha=\beta^2$, we have $(\beta^2)^2+\beta^2+1=\beta^4+\beta^2+1=0$

Thus $\mathrm{irr}(\beta,\mathbb{Z}_2[x])=\mathrm{irr}(1+\alpha,\mathbb{Z}_2[x])=x^4+x^2+1\in\mathbb{Z}_2[x]$

Grad student problems

p.273 #33 | Let E be extension of F, $\alpha\in E$ transcendental over F. Show all elements of $F(\alpha)$ that are not in F are transcendental over F.



Need to show all these are transcendental

Proof: Let $\beta\in F(\alpha)\setminus F$.

Since $\beta\in F(\alpha)=\{a+b\alpha : a,b\in F\}$, $\exists\, v,w\in F$ so that $\beta=v+w\alpha$.

Suppose for contradiction that β is not transcendental over F, i.e. β is algebraic over F.

This means there is a polynomial $p=a_0+a_1x+\dots+a_nx^n\in F[x]$ so that β is a root of p, i.e.

$a_0+a_1\beta+a_2\beta^2+\dots+a_n\beta^n=0$

But substitute in $\beta=v+w\alpha$ to get

$a_0+a_1(v+w\alpha)+a_2(v+w\alpha)^2+\dots+a_n(v+w\alpha)^n=0$

This expression can be rearranged to

$\underbrace{a_nw^n}_{c_n\in F}\alpha^n+\dots+\underbrace{(\text{terms with no }\alpha)}_{c_0\in F}=0$

$c_n\alpha^n+c_{n-1}\alpha^{n-1}+\dots+c_1\alpha+c_0=0$

But this can't be, because it means α is a root of $c_nx^n+\dots+c_1x+c_0\in F[x]$ which can't happen because α is transcendental over F, a contradiction! ∎