MTH 452 HW6

Sunday, March 31, 2024 8:51 PM

p. 252 # 1, 2, 16, 17, 24, 27

(grad students: p. 254 # 36

p. 252 | #1 | Prime & maximal ideals of $\mathbb{Z}_6$

- ideals of $\mathbb{Z}_6$ are subgroups of $\mathbb{Z}_6$ under $+$
- $\mathbb{Z}_6$ cyclic group $\Rightarrow$ all subgroups cyclic
- cyclic subgroups of $\mathbb{Z}_6$ are:

$\langle 0 \rangle = \{0\}$ ← trivial ideal

$\langle 1 \rangle = \mathbb{Z}_6$

$\langle 2 \rangle = \{0, 2, 4\}$

$\langle 3 \rangle = \{0, 3\}$ — all are ideals!

$\langle 4 \rangle = \{0, 4, 2\}$

$\langle 5 \rangle = \{0, 5, 4, 3, 2, 1\} = \mathbb{Z}_6$ ← nonproper ideal

$\Rightarrow$ four subgroups:

$\{0\}, \langle 2 \rangle, \langle 3 \rangle, \mathbb{Z}_6$

$$\begin{array}{ccccc} & & \mathbb{Z}_6 & & \\ & \cup\!\!| & & |\!\!\cup & \\ \langle 3 \rangle & & & & \langle 2 \rangle \\ & |\!\!\cup & & \cup\!\!| & \\ & & \{0\} & & \end{array}$$

maximal ideal (hence prime) → $\langle 3 \rangle$

$\langle 2 \rangle$ ← maximal ideal (hence prime)

$\{0\}$ ← not maximal ideal

#2 Prime + maximal ideals of $\mathbb{Z}_{12}$:

⊛ ideals of $\mathbb{Z}_{12}$ are subgroups of $\mathbb{Z}_{12}$ under $+$

⊛ $\mathbb{Z}_{12}$ cyclic $\Rightarrow$ subgroups cyclic

<u>cyclic subgroups of $\mathbb{Z}_{12}$</u>

$\langle 0 \rangle = \{0\}$

$\langle 1 \rangle = \mathbb{Z}_{12}$

isomorphic to

$\langle 2 \rangle = \{0, 2, 4, \dots, 10\} \cong \mathbb{Z}_6$

$\langle 3 \rangle = \{0, 3, 6, 9\} \cong \mathbb{Z}_4$

$\langle 4 \rangle = \{0, 4, 8\} \cong \mathbb{Z}_3$

since 5 rel prime to 12

$\langle 5 \rangle = \mathbb{Z}_{12}$

$\langle 6 \rangle = \{0, 6\} \cong \mathbb{Z}_2$

$\langle 7 \rangle = \mathbb{Z}_{12}$

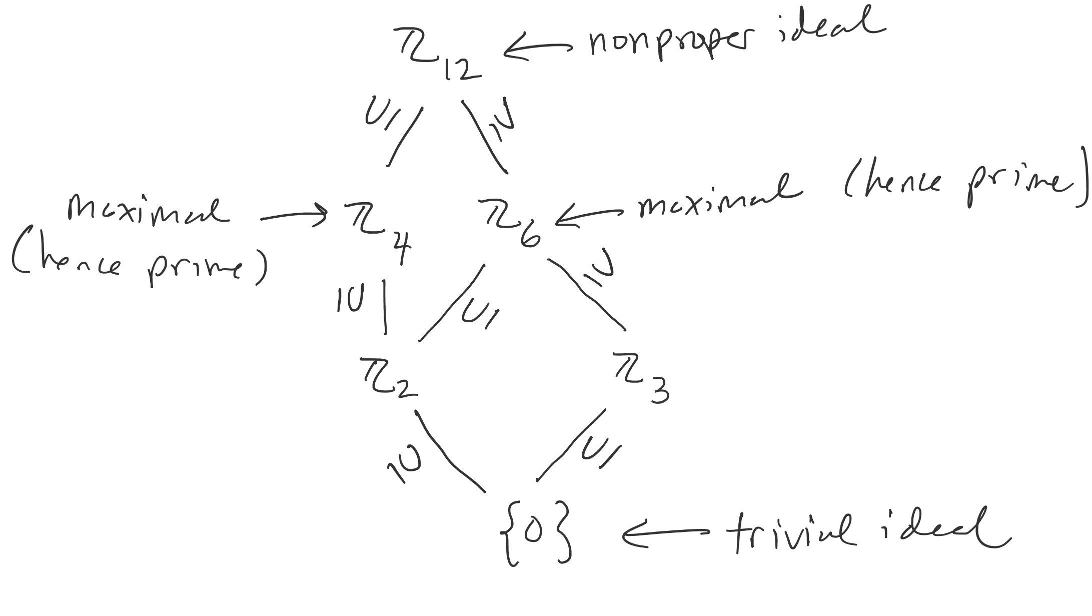
$\langle 8 \rangle = \{0, 8, 4\} \cong \mathbb{Z}_3$

$\langle 9 \rangle = \{0, 9, 6, 3\} \cong \mathbb{Z}_4$

$\langle 10 \rangle = \{0, 10, 8, 6, 4, 2\} \cong \mathbb{Z}_6$

$\langle 11 \rangle = \mathbb{Z}_{12}$

all these are ideals!



Find
#16/ Prime ideal of $\mathbb{Z}\times\mathbb{Z}$ that is not maximal

Consider: $\mathbb{Z}\times\{0\}$

① This is clearly closed under $\oplus$ and mult.

② $\mathbb{Z}\times\{0\}$ prime ideal b/c if

$$(x,y)(z,w)\in\mathbb{Z}\times\{0\},$$

then $\exists\, m\in\mathbb{Z}$ so that

$$(x,y)(z,w)=(m,0)$$

$$(xz,yw)=(m,0)$$

$$\Rightarrow \begin{cases} xz=m \\ yw=0 \longrightarrow y=0 \text{ or } w=0 \end{cases}$$

$$\Downarrow$$

$$(x,y)\in\mathbb{Z}\times\{0\} \text{ or } (z,w)\in\mathbb{Z}\times\{0\}$$

③ $\mathbb{Z}\times\{0\}$ not maximal because

$$\mathbb{Z}\times\{0\}\subseteq \underbrace{\mathbb{Z}\times 2\mathbb{Z}}_{\text{also an ideal!}} \subsetneq \mathbb{Z}\times\mathbb{Z}$$

#17/ Find nontrivial proper ideal of $\mathbb{Z}\times\mathbb{Z}$ that's not prime

Consider: $4\mathbb{Z}\times 8\mathbb{Z}$

① it's an ideal (check yourself!)

② not prime since

$$(2,4)(2,4)=(4,8)\in 4\mathbb{Z}\times 8\mathbb{Z}$$

but $(2,4)\notin 4\mathbb{Z}\times 8\mathbb{Z}$

#24 | Let R be finite cmt ring w/ unity. Show every prime ideal in R is maximal.

Thm 19.11 Every finite integral domain is a field.

Pf: Let N be a prime ideal in R.

Since N is prime ideal, R/N is an integral domain.

Since R is finite, R/N is ~~also~~ finite.

Thus by Thm 27.15, R/N is an integral domain.

But by Thm 19.11, we know that R/N is a field.

But Thm 27.9 shows that N is a maximal ideal. ■

Thm 27.9 Let R be cmt ring w/ unity. Then M is a maximal ideal of R iff R/M is a field.

Thm 27.15 R cmt w/ unity $N \neq R$ ideal. R/N integral domain iff N prime ideal in R

---

grad students

#36 | Let A, B ideals of cmt ring R.

Define The quotient $A:B = \{r \in R \mid rb \in A \text{ for all } b \in B\}$

Show $A:B$ is an ideal of R.

Pf: Let $x, y \in A:B$. Then $\forall b \in B$, $xb \in A$ and $yb \in A$.

Then since A is an ideal,

$$\underbrace{xb + yb}_{\in A \text{ since } A \text{ closed under } (+)} = (x+y)b \in A$$

$\in A$ since
A closed under $+$

Thus $A:B$ is closed under $+$.

Let $w \in R$. Then

$$w(A:B) = \{ wz \mid z \in A:B \}$$

If $y \in w(A:B)$ it means $y = wr$ so that $\forall b \in B$
$yb = wrb \in A$

Thus $y \in A:B$, so we have proved that $w(A:B) \subseteq A:B$.

Thus $A:B$ is an ideal, as was to be shown.