MTH 452 HW5

Sunday, March 31, 2024 5:27 PM

p. 218: #1, 2, 3, 4, 9, 10, 11, 14 ; p. 243 # 17, 26, 27, 30, 34, 37
(grad: p. 243 #19, 20, 29)

$\boxed{\text{p. 218 \#1}}$ $f = gq + r$

$$\left.\begin{aligned} f(x) &= x^6 + 3x^5 + 4x^2 - 3x + 2 \\ g(x) &= x^2 + 2x - 3 \end{aligned}\right\} \text{ in } \mathbb{Z}_7[x]$$

$$x^4 + x^3 - 5x^2 + 6x$$

$$x^2 + 2x - 3 \,\big)\, x^6 + 3x^5 + 4x^2 - 3x + 2$$

$$-(x^6 + 2x^5 - 3x^4)$$

$$x^5 - 3x^4 + 4x^2 - 3x + 2$$

$$-(x^5 + 2x^4 - 3x^3)$$

$$-5x^4 + 3x^3 + 4x^2 - 3x + 2$$

$$-(-5x^4 - 10x^3 + 15x^2)$$

$$13x^3 - 9x^2 - 3x + 2$$

mod 7

$$6x^3 - 2x^2 - 3x + 2$$

$$-(6x^3 + 12x^2 - 18x)$$

$$-14x^2 - 21x + 2$$

mod 7

$$0 - 0 + 2$$

$$\Longrightarrow \quad q(x) = x^4 + x^3 - 5x^2 + 6x$$

$$r(x) = 2$$

$\#4$ $\left.\begin{aligned} f(x) &= x^4 + 5x^3 - 3x^2 \\ g(x) &= 5x^2 - x + 2 \end{aligned}\right\}$ in $\mathbb{Z}_{11}[x]$

$9x^2 + 9x + 9$

$(5)(?) \overset{\text{mod } 11}{=} 1$

$g(x) = 5x^2 - x + 2$ (mod 11)

$$9x^2 + 8x + 9$$

$$5x^2 - x + 2 \,\overline{)\, \cancel{x^4} + 5x^3 - 3x^2}$$

$$-(\cancel{x^4} - 9x^3 + 7) \leftarrow \text{(in mod 11)}$$

$$-4x^3 - 10x^2$$

Mod 11

$$\cancel{7x^3} + x^2 \qquad 16 \text{ mod } 11 = 5$$

$$-(\cancel{7x^3} - 8x + 5)$$

$$\cancel{x^2} + 8x - 5$$

$$-(\cancel{x^2} - 9x + 7)$$

$$17x - 12$$

mod 11

$$6x + 10$$

$$(5)(?) = 1 \text{ mod}$$

$$\Downarrow$$

$$5(9) = 45 \text{ mod } 11 = 1$$

$$5(?) = 7 \text{ mod } 11$$

$$\downarrow$$

$$? = 8$$

$$5(8) = 40 \text{ mod } 11 = 7$$

x	$5x$ mod 11
0	0
1	5
2	10
3	4
4	9
5	3
6	9
7	2
8	7
9	1
10	6

$$\Rightarrow \begin{cases} q(x) = 9x^2 + 8x + 9 \\ r(x) = 17x - 12 \end{cases}$$

#9 Factor $x^4 + 4$ in $\mathbb{Z}_5[x]$:

x	$x^4 + 4$ mod 5
0	4
1	5 mod 5 = 0
2	20 mod 5 = 0
3	85 mod 5 = 0
4	256+4 = 260 mod 5 = 0

$$\Downarrow$$

$x-1$, $x-2$, $x-3$, and $x-4$ are factors

$$\Rightarrow x^4 + 4 = (x-1)(x-2)(x-3)(x-4)$$

#14) $f(x) = x^2 + 8x - 2 \in \mathbb{Q}[x]$

$$x^2 + 8x - 2 = 0$$

$\downarrow$ quadratic formula

$$x = \frac{-8 \pm \sqrt{64 - 4(1)(-2)}}{2}$$

$$= -4 \pm \tfrac{1}{2}\sqrt{72}$$

Since both roots involve $\sqrt{72} \notin \mathbb{Q}$, the polynomial $f \in \mathbb{Q}[x]$ is irreducible

If we regard $f \in \mathbb{R}[x]$ or $f \in \mathbb{C}[x]$, then it becomes reducible!

p. 243) #17) let $R = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$

and let

$$R' = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} : a, b \in \mathbb{Z} \right\}$$

show
① R' contains $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
② closed under $+$
③ closed under mult

Show
① R is a subring of $\mathbb{R}$
② R' is a subring of $M_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}$
③ $\begin{cases} \phi : R \to R' \\ \phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \end{cases}$ is an isomorphism

show
① R contains 1
② closed under $+$
③ closed under mult

Show R subring of $\mathbb{R}$ ① $1 = 1 + 0\sqrt{2}$ ✓

② $(a+b\sqrt{2}) \pm (c+d\sqrt{2})$

$= (a \pm c) + (b \pm d)\sqrt{2} \in R$

$\Rightarrow$ closed under $\pm$

③ $(a+b\sqrt{2})(c+d\sqrt{2}) = ac + ad\sqrt{2} + bc\sqrt{2} + 2bd$

$= (ac+2bd) + (ad+bc)\sqrt{2} \in R$

$\Rightarrow$ closed under mult

$\Rightarrow R$ is a subring of $\mathbb{R}$

Show R' subring of $M_2(\mathbb{Z})$:

① $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R'$ when $a=1, b=0$

② $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \pm \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & 2(b+d) \\ b+d & a+c \end{bmatrix} \in R'$

$a+c \in \mathbb{Z}$

$b+d \in \mathbb{Z}$

③ $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+2bd & 2ad+2bc \\ bc+ad & 2bd+ac \end{bmatrix}$

$= \begin{bmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{bmatrix} \in R'$

$ac+2bd \in \mathbb{Z}$

$ad+bc \in \mathbb{Z}$

$\Rightarrow R'$ subring of $M_2(\mathbb{Z})$

Isomorphism $\phi(a+b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$

① ϕ is 1-1:

$\phi(a+b\sqrt{2}) = \phi(c+d\sqrt{2})$

$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} = \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}$

$\Rightarrow \begin{matrix} a=c \\ b=d \end{matrix} \Rightarrow a+b\sqrt{2} = c+d\sqrt{2}$ ✓

② ϕ is onto

given $\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \in R'$, we know $a+b\sqrt{2} \in R$

maps to it, so ϕ is onto

③ structure-preserving

$\phi((a+\sqrt{2}b)+(c+d\sqrt{2})) = \phi((a+c)+(d+b)\sqrt{2})$

$= \begin{bmatrix} a+c & 2(d+b) \\ d+b & a+c \end{bmatrix}$

$= \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}$

$= \phi(a+b\sqrt{2}) + \phi(c+d\sqrt{2})$

and

$\phi((a+\sqrt{2}b)(c+d\sqrt{2})) = \phi(ac + (ad+bc)\sqrt{2} + 2bd)$

$$= \phi\left((ac+2bd)+(ad+bc)\sqrt{2}\right)$$

$$= \begin{bmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{bmatrix}$$

these are equal!

while

$$\phi(a+b\sqrt{2})\,\phi(c+d\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\begin{bmatrix} c & 2d \\ d & c \end{bmatrix}$$

$$= \begin{bmatrix} ac+2bd & 2(ad+bc) \\ bc+ad & 2bd+ac \end{bmatrix}$$

$\Rightarrow \phi$ is an isomaphism!

#26 Let R be cmt ring, let $a \in R$.

Show $I_a := \{x \in R : ax = 0\}$ is an ideal of R

<u>Soln</u>: if $\alpha, \beta \in I_a$, it means $a\alpha = 0$ and $a\beta = 0$

So

$$\Downarrow$$

$$a\alpha \pm a\beta = 0 \pm 0$$

$$a(\alpha \pm \beta) = 0$$

$$\Downarrow$$

$$\alpha \pm \beta \in I_a$$

$\Rightarrow I_a$ closed under $+$

If $\gamma \in R$, then

$\gamma I_a = \{ \gamma x : x \in R, ax = 0 \}$

But since any $f \in \gamma I_a$ has property that $\exists x^* \in R$ so that $ax^* = 0$ and $f = \gamma x^*$.

But then $af = a\gamma x^* \underset{\substack{\uparrow \\ R \text{ commutative}}}{=} \underbrace{ax^*}_{=0}\gamma = 0$

Thus if $f \in \gamma I_a$, then $f \in I_a$.

$\Rightarrow$ So, $\gamma I_a \subseteq I_a$, hence I_a absorbs multiplication.

Thus I_a is an ideal.

#27) Show $\cap$ of ideals is an ideal.

Soln: Let N_1, N_2 be ideals of a ring R and define $N = N_1 \cap N_2$.

closed under $\oplus$

If $a, b \in N$, then $a, b \in N_1$ and $a, b \in N_2$.

Since N_1, N_2 are ideals, $a \pm b \in N_1$ and $a \pm b \in N_2$

$\Rightarrow a \pm b \in N = N_1 \cap N_2$

absorb mult

If $w \in R$, then since N_1 and N_2 are ideals,

$wN_1 \subseteq N_1$ and $wN_2 \subseteq N_2$.

Thus $wN \subseteq N$, so N absorbs mult.

Thus N is an ideal. ▲

#28) ... all nilpotent elts in a comt. ring R

#30) Show collection of all nilpotent elts in a comm. ring R forms an ideal (the nilradical, $\boxed{\text{nil}(R)}$)

$\boxed{\exists n \in \mathbb{Z}^+ \text{ s.t. } a^n = 0}$

Soln: Let $x, y \in \text{nil}(R)$, meaning that $\exists m, n \in \mathbb{Z}^+$ so that $x^n = 0$ and $y^m = 0$. Clearly, $-x, -y \in \text{nil}(R)$, since $(-x)^n = (-1)^n x^n = 0$, $(-y)^m = (-1)^m y^m = 0$.

So, Consider

$$(x+y)^{m+n} \underset{\text{binomial thm}}{=} \sum_{k=0}^{m+n} \binom{m+n}{k} x^k y^{m+n-k}$$

since $m+n > m$; 0 since $m+n-1 > m$

$$= \binom{m+n}{0} x^0 \underbrace{\cancel{y^{m+n}}}_{=0} + \binom{m+n}{1} \cancel{x y^{m+n-1}}$$

$$+ \cdots + \binom{m+n}{n} \underbrace{\cancel{x^n y^m}}_{=0} + \binom{m+n}{n+1} \underbrace{\cancel{x^{n+1}} y^{m-1}}_{=0 \text{ since } n+1 \geq n}$$

$$+ \cdots + \binom{m+n}{m+n} \underbrace{\cancel{x^{m+n}}}_{=0}$$

$$= 0$$

Thus $x+y \in \text{nil}(R)$.

Now let $r \in R$. Compute

$$r(\text{nil}(R)) = \{rx : x \in \text{nil}(R)\}$$

Let $w \in r(\text{nil}(R))$, then $w = rx$ for some $x \in \text{nil}(R)$.

Since $x \in \text{nil}(R)$, $\exists n \in \mathbb{Z}^+$ s.t. $x^n = 0$.

Thus

$\ldots$ since $x^n = 0$

Thus

$$w^n = (rx)^n = r^n x^n = 0 \quad \text{since } x^n = 0.$$

Thus $r \in \mathrm{nil}(R)$.

So we have $r(\mathrm{nil}(R)) \subseteq \mathrm{nil}(R)$,

so $\mathrm{nil}(R)$ absorbs multiplication.

Thus $\mathrm{nil}(R)$ is an ideal of R.

#34 Let R be a cmt ring and let N be an ideal of R.

Show set $\sqrt{N} := \{a \in R : \exists n \in \mathbb{Z}^+ \; a^n \in N\}$ is an ideal of R.

called radical of N

Solu: Let $x, y \in \sqrt{N}$, so $\exists m, n \in \mathbb{Z}^+$ so that $x^m \in N$, $y^n \in N$

Then,

$$(x+y)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k y^{m+n-k}$$

$$= \binom{m+n}{0} x^0 \underbrace{y^{m+n}}_{\in N} + \binom{m+n}{1} x \underbrace{y^{m+n-1}}_{\in N} + \cdots + \binom{m+n}{n} \underbrace{x^m}_{} \underbrace{y^n}_{\in N}$$

$$+ \binom{m+n}{m+1} \underbrace{x^{m+1}}_{\in N} y^{n-1} + \cdots + \binom{m+n}{m+n} \underbrace{x^{m+n}}_{\in N} y^0$$

$\Rightarrow$ Since each term contains a factor in N and N is an ideal (hence absorbs mult)

$\Rightarrow$ Since
in N and N is an ideal (hence absorbs mult)
we see ~~e~~ each term lies in N

Since N is an ideal, it is closed under $\oplus$,
so $(x+y)^{m+n} \in N$

$\Rightarrow x+y \in \sqrt{N}$

(Similarly, can argue that $(-x)^m = (-1)^m x^m \in N$ so $\oplus$ inverses are taken care of)

Now if $w \in R$, then

$$w\sqrt{N} := \{wx : \exists n \in \mathbb{Z}^+ \ x^n \in N\}$$

Let $z \in w\sqrt{N}$, so $z = wx$ and $x^n \in N$
But then $z^n = (wx)^n = w^n x^n$ and since N absorbs multiplication, we have $x^n \in N \Rightarrow z^n \in \sqrt{N}$.

$\Rightarrow w\sqrt{N} \subseteq \sqrt{N}$

Good students

P.243 | #20 | R cmt w/ unity prime chr p

Show: $\begin{cases} \phi_p : R \to R \\ \phi_p(a) = a^p \end{cases}$ is homomorphism

Pf: $\phi_p(a+b) = (a+b)^p = \sum_{k=0}^{p} \binom{p}{k} a^k b^{p-k}$

$\binom{p}{k} = \frac{p!}{k!(p-k)!}$

$$= \binom{p}{0} b^p + \left[\sum_{k=1}^{p-1} \binom{p}{k} a^k b^{p-k}\right] + \binom{p}{p} a^p$$

$$= b^p + \left[p! \sum_{k=1}^{p-1} \frac{a^k b^{p-k}}{(p-k)!\,k!}\right] + a^p$$

$k!(p-k)!$

$$= b^p + \underbrace{\left[p! \sum_{k=1}^{p-1} \frac{a^k b^{p-k}}{(p-k)!\,k!} \right]}_{\substack{=0 \text{ because} \\ R \text{ has characteristic } p \\ (\text{it has factor of } p)}} + a^p$$

$$= b^p + a^p$$

$$= \phi_p(b) + \phi_p(a)$$

$$\phi_p(ab) = (ab)^p = a^p b^p = \phi_p(a)\phi_p(b)$$

Thus ϕ_p is a homomorphism.