MTH 452 HW4

Sunday, February 25, 2024 11:17 AM

p.196 # 1,2 ; p.207 # 1,2,3,4,5,6,7,8,9,10,12,13,14,15,20,21,24,27

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p.196 **#1)** $D = \{n+mi : n,m \in \mathbb{Z}, i^2=-1\}$

Let $X = \{(a,b) : a,b \in D\}$

Define $(a,b) \sim (c,d) \longleftrightarrow ad = bc$

$\frac{a}{b} = \frac{c}{d}$

$ad = bc$

$a = a_1 + ia_2$
$b = b_1 + ib_2$
$c = c_1 + ic_2$
$d = d_1 + id_2$

$(a_1+ia_2)(d_1+id_2) = (b_1+ib_2)(c_1+ic_2)$

defined as

$[(a,b)] := \{(x,y) : (x,y) \sim (a,b)\}$

Thus, $F = \{[(a,b)] : (a,b) \in X\}$

(for example $[(1+i, 2-i)] \in F$ refers to all fractions equal to $\frac{1+i}{2-i}$, so things like $\frac{2+2i}{4-2i}$, etc)

#2) $D = \{m + n\sqrt{2} : m,n \in \mathbb{Z}\}$

Let $X = \{(a,b) : a,b \in D\}$

so

$F = \{[(a,b)] : (a,b) \in X\}$

(for ex, $[(2-\sqrt{2}, 5+6\sqrt{2})] \in F$ refers to all fractions like
$\frac{2-\sqrt{2}}{5+6\sqrt{2}} = \frac{4-2\sqrt{2}}{10+12\sqrt{2}} = \dots$)

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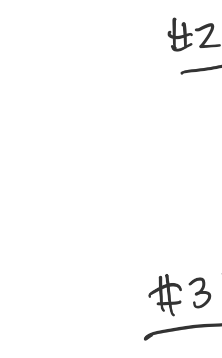
p.207 **#1)** $f(x) = 4x-5$, $g(x) = 2x^2-4x+2$ in $\mathbb{Z}_8[x]$



$\Downarrow$

$f(x)g(x) = (4x-5)(2x^2-4x+2) \mod 8$

$= 8x^3 - 26x^2 + 28x - 10 \mod 8$

$= 0 \quad 6 \quad 4 \quad 6$

$= 6x^2 + 4x + 6$

#2) $(x+1)(x+1) \mod 2$

$= x^2 + 2x + 1 \mod 2$

$= x^2 + 1$

#3) $(2x^2+3x+4)(3x^2+2x+4) \mod 5$

$= 6x^4 + 4x^3 + 8x^2 + 9x^3 + 6x^2 + 12x + 12x^2 + 8x + 16 \mod 5$

$= 6x^4 + 13x^3 + 26x^2 + 20x + 16 \mod 5$ (20x crossed out, 0)

$= x^4 + 3x^3 + 6x^2 + 1$

#6) polynomials of degree ≤ 2 in $\mathbb{Z}_5$:

$= \{ax^2 + bx + c : a,b,c \in \{0,1,2,3,4\}\}$

5 choices

$\Rightarrow 5^3 = 125$ such polynomials

#7) $\phi_2(x^2+3) = 2^2 + 3 = 4+3 = 7$
(in $\mathbb{C}$)

#9) (in $\mathbb{Z}_7$)

$\phi_3\left((x^4+2x)(x^3-3x^2+3)\right)$

$= (3^4+6)(3^3 - 3(3^2)+3) \mod 7$

$= (87)(27-27+3) \mod 7$

$= 261 \mod 7$

$= 2$

$$\begin{array}{r} 37 \\ 7\overline{)261} \\ -21 \\ \hline 51 \\ 49 \\ \hline 2 \end{array}$$

13
14
15
20
21
24
27

#12) Find all zeros of $x^2 + 1 \in \mathbb{Z}_2[x]$

Soln:

x	x^2+1
0	1
1	$1+1 = 2 \mod 2 = 0$

$\Downarrow$

$\boxed{x=1}$

#13) $x^3 + 2x + 2 \in \mathbb{Z}_7[x]$

x	
0	2
1	5
2	$8+4+2 = 14 \mod 7 = 0$
3	$27+6+2 = 35 \mod 7 = 0$
4	$64+8+2 = 74 \mod 7 = 4$
5	$125+10+2 = 137 \mod 7 = 4$
6	$216+12+2 = 230 \mod 7 = 6$

$\Downarrow$

zeros are $\boxed{x = 2, 0}$

#21) $\phi_5 : \mathbb{Q}[x] \to \mathbb{R}$ ← eval homomorphism
Find 6 elts of $Ker(\phi_5)$

$x-5,\ x^2-5x,\ x^3-5x^2,\ x^4-5x^3,\ x^5-5x^4,\ x^6-5x^6$

mult by x   mult x

#24) Let D be an int dom.

Spz $D[x]$ is not an int dom.

Then $\exists\, f,g \in D[x]$ so that $f = a_0 + a_1x + \dots + a_nx^n$
$g = b_0 + b_1x + \dots + b_mx^m$

and $fg = a_0b_0 + a_1b_1x + \dots + \text{last term} = 0$

Only way this equals zero is if

$a_0b_0 = 0,\ a_1b_1 = 0, \dots$

But we can't have $a_0b_0 = 0$ since $a_0, b_0 \in D$ and D is an integral domain, so that's a contradiction.

Thus $D[x]$ is an integral domain. ☒

#27) $chr(F) = 0$

$D(a_0 + a_1x + \dots + a_nx^n) = a_1 + 2a_2x + \dots + na_nx^{n-1}$

(a) Show D is homomorphism of $\langle F,+\rangle$ to $\langle F,+\rangle$

Let $f = \sum_{k=0}^{\infty} a_kx^k$, $g = \sum_{k=0}^{\infty} b_kx^k$

Then

$D(f+g) = D\left(\sum_{k=0}^{\infty}(a_k+b_k)x^k\right)$

$= \sum_{k=1}^{\infty} k(a_k+b_k)x^{k-1}$

$= \sum_{k=1}^{\infty} ka_kx^{k-1} + \sum_{k=1}^{\infty} kb_kx^{k-1}$

$= D(f) + D(g)$,

completes the proof that D is structure-preserving, hence D is a homomorphism (of the group structure).

Clearly, D is not a ring homomorphism because

$x(x+1) = x^2 + x$
$\downarrow D$
$2x+1$

But

$D(x)D(x+1) = 1 \cdot 1 = 0$

So,

$D(x)D(x+1) \neq D(x(x+1))$,

showing D is not a homomorphism on the multiplication structure.

(b) $Ker(D) = \{f \in F[x] : D(f) = 0\}$

if $f \in F[x]$ with $f = a_0 + a_1x + \dots + a_nx^n$, then

$D(f) = 0$

whenever

$a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} = 0$

which happens whenever

$a_1 = a_2 = \dots = a_n = 0$

In other words, the only $f \in F[x]$ so that $D(f) = 0$ are those f so that

$f = a_0 + 0 + 0 + 0 \dots + 0 = a_0$,

i.e. the constant polynomials

$Ker(D) = \{f \in F[x] : f(x) = \text{constant}\}$