HW6 MTH 450/55- Fall 2023

Saturday, October 14, 2023 12:08 AM

p.55 #21 (a) $\langle 25\mathbb{Z}, + \rangle$

five elts: $0, 5, 10, 20, 25$

(b) $\langle \{ (\frac{1}{2})^n \mid n \in \mathbb{Z} \}, \cdot \rangle$

five elts: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

(c) $\langle \{ \pi^n \mid n \in \mathbb{Z} \}, \cdot \rangle$

five elts: $1, \pi, \pi^2, \pi^3, \pi^4$

#22 $X = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \longrightarrow X^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ because $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$X^2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$\vdots$

So, $\langle X \rangle = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$

#25 $X = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \longrightarrow X^{-1} = \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix}$

$$X^{-2} = \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}$$

$$X^{-3} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/8 \\ -1/8 & 0 \end{bmatrix}$$

$\vdots$

$$X^2 = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$X^3 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ -8 & 0 \end{bmatrix}$$

$\vdots$

can show by induction (see HW5 #23 where I did it for a similar prob)

$$X^n = \begin{cases} \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}, & n \text{ even} \\ \begin{bmatrix} 0 & -2^n \\ -2^n & 0 \end{bmatrix}, & n \text{ odd} \end{cases} \qquad X^{-n} = \begin{cases} \begin{bmatrix} 1/2^n & 0 \\ 0 & 1/2^n \end{bmatrix}, & n \text{ even} \\ \begin{bmatrix} 0 & -1/2^n \\ -1/2^n & 0 \end{bmatrix}, & n \text{ odd} \end{cases}$$

$= 2^{-n}$

Thus we have

$$\langle X \rangle = \left\{ \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix} : n \in \mathbb{Z}, n \text{ even} \right\} \cup \left\{ \begin{bmatrix} 0 & 2^n \\ 2^n & 0 \end{bmatrix} : \begin{matrix} n \in \mathbb{Z} \\ n \text{ odd} \end{matrix} \right\}$$

#33 $X = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$X = X^3 \ldots$

$X^2 = X^4 \ldots$

$$X^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{identity}}$$

$$X^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\vdots$

So,

$$\langle X \rangle = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right\}$$

#34 similar...

$X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ call it A

$X^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ call it B

$X^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ call it C

$X^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ call it D

A, B, C, D

$X^5 = A$

Therefore, $\langle X \rangle = \{A, B, C, D\}$

p.66 #1 $n = 42, m = 9$

$\hookrightarrow$ find q, r so that $\begin{cases} n = mq + r \\ 0 \le r < m \end{cases}$

$9\overline{)42}$, -36, 6 $\rightarrow$ 4 R6 (q, r)

So, $\begin{pmatrix} r = 6 \\ q = 4 \end{pmatrix}$

$42 = 9 \cdot 4 + 6$

$36 + 6 = 42$ ✓

#2 $n = -42, m = 9$

$$\underbrace{-42 = -9 \cdot 4 - 6}_{-36 - 6 = -42\ ✓} \quad \begin{pmatrix} q = -4 \\ r = -6 \end{pmatrix}$$

#3 $n = -50, m = 8$

$8\overline{)50}$, -48, 2 $\rightarrow$ 6 R2 $\rightarrow$ $50 = 6 \cdot 8 + 2$

$-50 = (-6) \cdot 8 - 2 \rightarrow \begin{cases} q = -6 \\ r = -2 \end{cases}$

#4 $n = 50, m = 8$

$50 = 6 \cdot 8 + 2 \longrightarrow \begin{cases} q = 6 \\ r = 2 \end{cases}$