# HW10 MTH450/550 Fall 2023

Sunday, November 12, 2023 6:08 PM

**p.101 #6** subgroup $\{\rho_0, \mu_2\}$ of $D_4$ → (from table 8.12)

find left cosets:

Call $X = \{\rho_0, \mu_2\}$

$\rho_0 X = \{\rho_0\rho_0, \rho_0\mu_2\} = \{\rho_0, \mu_2\}$

$\rho_1 X = \{\rho_1\rho_0, \rho_1\mu_2\} = \{\rho_1, \delta_2\}$

$\rho_2 X = \{\rho_2\rho_0, \rho_2\mu_2\} = \{\rho_2, \mu_1\}$

$\rho_3 X = \{\rho_3\rho_0, \rho_3\mu_2\} = \{\rho_3, \delta_1\}$

$\mu_1 X = \{\mu_1\rho_0, \mu_1\mu_2\} = \{\mu_1, \rho_2\}$

$\mu_2 X = \{\mu_2\rho_0, \mu_2\mu_2\} = \{\mu_2, \rho_0\}$

$\delta_1 X = \{\delta_1\rho_0, \delta_1\mu_2\} = \{\delta_1, \rho_3\}$

$\delta_2 X = \{\delta_2\rho_0, \delta_2\mu_2\} = \{\delta_2, \rho_1\}$

**#28** Let H be a subgrp of G such that $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$. Show every left coset $gH$ is same as the right coset $Hg$.

Proof: From $\forall g \in G\ \forall h \in H\ \ g^{-1}hg \in H$

$\Updownarrow$

$\forall g \in G\ \forall h \in H\ \ gg^{-1}hg \in gH$

$\Updownarrow$

$\forall g \in G\ \forall h \in H\ \ hg \in gH$

But $hg \in Hg$. So we have shown $\forall g \in G\ \forall h \in H\ \ gH = Hg$

**p.110 #1** $\mathbb{Z}_2 \times \mathbb{Z}_4 = \left\{ \begin{matrix} (0,0), (0,1), (0,2), (0,3), \\ (1,0), (1,1), (1,2), (1,3) \end{matrix} \right\}$

$|(0,0)| = 1$

$|(0,1)|$

↳ $(0,1)^2 = (0,1) + (0,1) = (0,2)$

↳ $(0,1)^3 = (0,2) + (0,1) = (0,3)$

↳ $(0,1)^4 = (0,3) + (0,1) = (0,0)$ ✓

$\Rightarrow |(0,1)| = 4$

$|(0,2)| = 2$
↳ $(0,2) \to (0,0)$

$|(0,3)| = 4$
↳ $(0,3) \to (0,2) \to (0,1) \to (0,0)$

$|(1,0)| = 2$
↳ $(1,0) \to (0,0)$

$|(1,1)| = 4$
↳ $(1,1) \to (0,2) \to (1,3) \to (0,0)$

$|(1,2)| = 2$
↳ $(1,2) \to (0,0)$

$|(1,3)| = 4$
↳ $(1,3) \to (0,2) \to (1,1) \to (0,0)$

**#3** order of $(2,6)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12}$:

$(2,6) \to (0,0) \Rightarrow |(2,6)| = 2$

**#4** order of $(2,3)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{15}$

$(2,3) \to (4,6) \to (0,9) \to (2,12) \to (4,0)$

$\downarrow$

$(2,0) \leftarrow (0,12) \leftarrow (4,9) \leftarrow (2,6) \leftarrow (0,3)$

$\downarrow$

$(4,3) \to (0,6) \to (2,9) \to (0,12) \to (2,0)$

$\downarrow$

$(0,0) \leftarrow (4,12) \leftarrow (2,9) \leftarrow (0,6) \leftarrow (4,3)$

$\Rightarrow |(2,3)| = 20$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12}$

**#9** Proper nontrivial subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$

$\langle(0,0)\rangle = \{(0,0)\}$ (trivial)

$\langle(0,1)\rangle = \{(0,1), (0,0)\}$

$\langle(1,0)\rangle = \{(1,0), (0,0)\}$

$\langle(1,1)\rangle = \{(1,1), (0,0)\}$

**#16** isomorphic

By Fund thm, $\mathbb{Z}_{12} = \mathbb{Z}_4 \times \mathbb{Z}_3$, so

$\mathbb{Z}_2 \times \mathbb{Z}_{12} = \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3$

But also by fund thm, $\mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$, so

$\mathbb{Z}_4 \times \mathbb{Z}_6 = \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2$

The order of factors doesn't matter.

Thm 11.12 shows these representations are unique, completing the proof.

**#46** Let $\langle G_1, *_1\rangle$ and $\langle G_2, *_2\rangle$ be abelian groups. Let $G = G_1 \times G_2$.

Let $(a,b), (c,d) \in G$. Compute for group $\langle G, *\rangle$:

$(a,b) * (c,d) = (a *_1 c, b *_2 d)$

$\overset{\text{abelian assumption}}{=} (c *_1 a, d *_2 b)$

$= (c,d) * (a,b)$,

completing the proof. ∎

**#54** Let $G, H, K$ be finitely generated abelian groups. Show if $G \times K$ is isomorphic to $H \times K$, then $G$ is isomorphic to $H$.

Pf: Let $\phi: G \times K \to H \times K$ be an isomorphism.

Consider the function

$\psi: G \to H$ given by

$\psi(g) = 1^{st}$ component of $\phi(g, e_2)$, where $e_2$ is the identity of $H$.

① Since $\phi$ is 1-1, so is $\psi$: Suppose $\psi(g_1) = \psi(g_2)$

$\Rightarrow$ Then since $\phi((g_1, e_2)) = \phi((g_2, e_2)) \Rightarrow (g_1, e_2) = (g_2, e_2)$, we have that $g_1 = g_2$

② Since $\phi$ is onto, so is $\psi$: let $w \in H$. Then is some $(x,y) \in G \times K$ so that

$\phi((x,y)) = (w, z)$ for some $z$

Then $x \in G$ with $\psi(x) = w$. So $\psi$ is onto.

③ Suppose operation of $G$ is $*_G$, op of $H$ is $*_H$, and op of $K$ is $*_K$.

Since $\phi((a,b) *_{G\times K} (c,d)) = \phi(a,b) *_{H\times K} \phi(c,d) = (a' *_H c', b' *_K d')$

$\quad\ \ \| \qquad\qquad\qquad\qquad\qquad \uparrow \qquad\qquad \uparrow$

$\phi(a *_G c, b *_K d)$ call $(a', b')$ call $(c', d')$

We see that $\quad a' = \psi(a) \qquad c' = \psi(c)$

$\psi(a *_G c) = \psi(a) *_H \psi(c)$,

showing $\psi$ is structure preserving.

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For grad students

**p.101 #43** $H, K$ subgroups of $G$

Define $\sim$ on $G$ by $a \sim b$ iff $a = hbk$ for some $h \in H$ and $k \in K$.

(a) Show $\sim$ is equiv relation.

Pf

① Reflexive

any $x \in G$,

$x = xx^{-1}x$, so $x \sim x$

$\uparrow \quad \uparrow\ \uparrow\ \uparrow$

$a \quad h\ b\ k$

② Symmetric

if $x \sim y$, then $\exists h \in H, k \in K$ so that

$x = hyk$

$\Rightarrow h^{-1}xk^{-1} = y$

Choose $h^* = h^{-1} \in H$ and $k^* = k^{-1} \in K$ (which exist b/c $H$ and $K$ are groups)

Then $y = h^* x k^*$ for $h^* \in H$ and $k^* \in K$

$\Rightarrow y \sim x$

③ Transitive

if $x \sim y$ and $y \sim z$, then $\exists h_1, h_2 \in H$, $k_1, k_2 \in K$ so that

$x = h_1 y k_1$ and $y = h_2 z k_2$

Plug eqt for $y$ into eqt for $x$ to get

$x = h_1 (h_2 z k_2) k_1$

$= (h_1 h_2) z (k_1 k_2)$

But since $H$ and $K$ are groups, $\underbrace{h_1 h_2}_{=: h^*} \in H$ and $\underbrace{k_1 k_2}_{=: k^*} \in K$.

So we have

$x = h^* z k^*$, thus $x \sim z$

**p.112 #39** Let $G$ be abelian. Show $X = \{g \in G : g \text{ is of finite order}\}$ is a subgroup of $G$.

① if $a, b \in X$, then $\exists m, n$ so that $a^m = e$ and $b^n = e$.

Thus for all $j \in \{0, 1, 2, 3, \dots\}$,

$(a^m)^j = e^j = e$ and $(b^n)^j = e^j = e$

Let $l = \text{lcm}(m,n)$. Then

$(ab)^l = (ab)(ab)(ab)\cdots(ab)$

$\overset{(G \text{ abelian})}{=} a^l b^l = ee = e$

So $ab$ is finite order $\longrightarrow ab \in X \Rightarrow$ closed under operation

② $e^j = e$ for any $j \Rightarrow e \in X \Rightarrow$ contains identity

③ $a \in X \to \exists j\ \ a^j = e \Rightarrow e = a^{-j} = (a^{-1})^j \Rightarrow a^{-1} \in X$

$\uparrow$ mult by $a^{-1}$ $j$ times

$\Rightarrow$ closed under inverses

$\Rightarrow X$ is a subgroup of $G$

**#41** torsion subgroup of $\mathbb{R}^*$

if $x > 1$: $x^n \to \infty$ as $x \to \infty \Rightarrow x$ has infinite order

if $x < -1$: $x^n$ oscillates in growing size $\Rightarrow x$ has inf order

if $-1 < x < 1$: $x^n \to 0$, ~~but~~ never reaches it in finite steps $\Rightarrow x$ has $\infty$ order

$\Rightarrow \{-1, 1\}$ is the torsion subgroup

↑ both have finite order in $\mathbb{R}^*$