MTH 229H - EXAM 1 FALL 2023 SOLUTION

Friday, 14 September
Instructor: Tom Cuchta

Instructions:

- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (12 points) Find all solutions of the given equation.

(a) (3 points) $6x = 2x + 1$

Solution: Subtract $2x$ from both sides to get $4x = 1$, so $x = \dfrac{1}{4}$.

(b) (3 points) $x^2 + x - 12 = 0$ *Solution*: The left-hand side factors yielding $(x+4)(x-3) = 0$, hence $x + 4 = 0$ OR $x - 3 = 0$. Thus $x = -4, 3$.

(c) (3 points) $x^2 + 5x - 1 = 0$ *Solution*: In this case, it is not obvious if it factors or not. So we use the quadratic formula:

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(-1)}}{2} = -\frac{5}{2} \pm \frac{\sqrt{25+4}}{2} = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}.$$

(d) (3 points) $3^{5x-2} = 11^{-x+1}$

Solution: Take ln of both sides to get

$$\ln\left(3^{5x-2}\right) = \ln\left(11^{-x+1}\right).$$

Using the property of logarithms $\ln(a^b) = b\ln(a)$, we obtain

$$(5x-2)\ln(3) = (-x+1)\ln(11).$$

Adding $x\ln(11)$ and $2\ln(3)$ to both sides yields

$$\Big(5\ln(3) + \ln(11)\Big)x = \ln(11) + 2\ln(3).$$

Therefore,

$$x = \frac{\ln(11) + 2\ln(3)}{5\ln(3) + \ln(11)}.$$

2. (8 points) Find the exact value

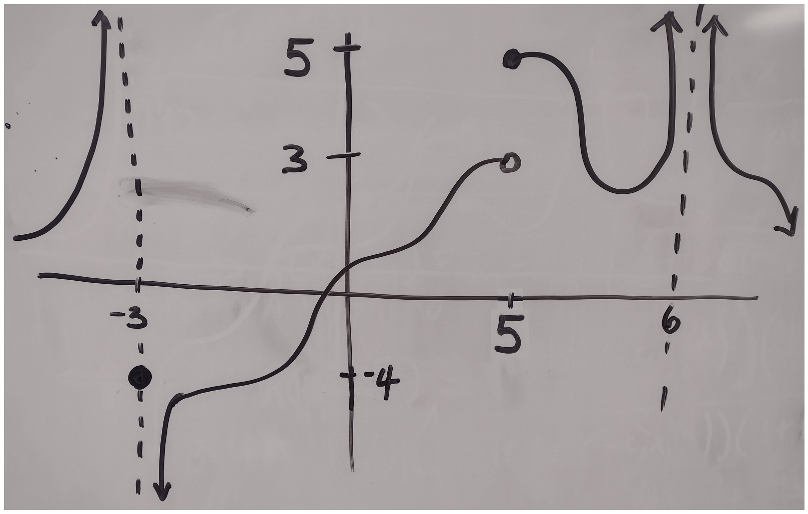
(a) (2 points) $\cos(\pi) = -1$

(b) (2 points) $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

(c) (2 points) $\sin\left(\dfrac{4\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$

(d) (2 points) $\tan\left(\dfrac{3\pi}{4}\right) = \dfrac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)} = \dfrac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$

3. (36 points) Consider the following graph of a function named f:



Use the graph to resolve the following limits and function values (note: sometimes these may not exist! make sure you state that, if so!):

(a) (3 points) $\lim_{t\to -3^-} f(t) = +\infty$

(b) (3 points) $\lim_{t\to -3^+} f(t) = -\infty$

(c) (3 points) $\lim_{t\to -3} f(t) = \text{DNE}$

(d) (3 points) $f(-3) = -4$

(e) (3 points) $\lim_{t\to 5^-} f(t) = 3$

(f) (3 points) $\lim_{t\to 5^+} f(t) = 5$

(g) (3 points) $\lim_{t\to 5} f(t) = \text{DNE}$

(h) (3 points) $f(5) = 5$

(i) (3 points) $\lim_{t\to 6^-} f(t) = +\infty$

(j) (3 points) $\lim_{t\to 6^+} f(t) = +\infty$

(k) (3 points) $\lim_{t\to 6} f(t) = +\infty$

(l) (3 points) $f(6) = \text{DNE}$

4. (40 points) Compute the limit. These all exist, so they resolve to a finite number!

(a) (8 points) $\lim_{x \to 2} \frac{x^2 - 7x + 1}{x^2 + x - 10}$

Solution: In this case, we can just substitute in $x = 2$:

$$\begin{aligned} \lim_{x \to 2} \frac{x^2 - 7x + 1}{x^2 + x - 10} &= \frac{2^2 - 7(2) + 1}{2^2 + 2 - 10} \\ &= \frac{4 - 14 + 1}{4 + 2 - 10} \\ &= \frac{-9}{-4} \\ &= \frac{9}{4}. \end{aligned}$$

(b) (8 points) $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + 3x - 4}$

Solution: In this case, substituting in $x = 1$ yields $\frac{0}{0}$, meaning there is more work to do. So, we factor:

$$x^2 + x - 2 = (x + 2)(x - 1),$$

and

$$x^2 + 3x - 4 = (x + 4)(x - 1).$$

Therefore we observe that

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + 3x - 4} &= \lim_{x \to 1} \frac{(x + 2)(x - 1)}{(x + 4)(x - 1)} \\ &= \lim_{x \to 1} \frac{x + 2}{x + 4} \\ &= \frac{1 + 2}{1 + 4} \\ &= \frac{3}{5}. \end{aligned}$$

(c) (8 points) $\lim_{x\to 4} \frac{\sqrt{3x+4}-4}{x-4}$

Solution: Substituting in $x = 4$ yields $\frac{0}{0}$, meaning there is more work to do. So, we will multiply the function by a "convenient form of one" using the "algebraic conjugate" of the numerator, i.e.

$$\begin{aligned}
\lim_{x\to 4} \frac{\sqrt{3x+4}-4}{x-4} &= \lim_{x\to 4} \left(\frac{\sqrt{3x+4}-4}{x-4}\right)\left(\frac{\sqrt{3x+4}+4}{\sqrt{3x+4}+4}\right) \\
&= \lim_{x\to 4} \frac{3x+4-16}{(x-4)\left(\sqrt{3x+4}+4\right)} \\
&= \lim_{x\to 4} \frac{3x-12}{(x-4)\left(\sqrt{3x+4}+4\right)} \\
&= \lim_{x\to 4} \frac{3\cancel{(x-4)}}{\cancel{(x-4)}\left(\sqrt{3x+4}+4\right)} \\
&= \lim_{x\to 4} \frac{3}{\sqrt{3x+4}+4} \\
&= \frac{3}{\sqrt{12+4}+4} \\
&= \frac{3}{\sqrt{16}+4} \\
&= \frac{3}{8}.
\end{aligned}$$

(d) (8 points) $\lim_{x\to 0} \frac{\sin(5x)}{\sin(2x)}$

Solution: Substituting $x = 0$ yields $\frac{0}{0}$. So we recall the limit $\lim_{x\to 0} \frac{\sin(x)}{x}$ and compute

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(5x)}{\sin(2x)} &= \lim_{x\to 0} \left(\frac{\sin(5x)}{\sin(2x)}\right)\left(\frac{5x}{5x}\right)\left(\frac{2x}{2x}\right) \\
&= \underbrace{\left(\lim_{x\to 0} \frac{\sin(5x)}{5x}\right)}_{=1} \underbrace{\left(\lim_{x\to 0} \frac{2x}{\sin(2x)}\right)}_{=1} \left(\lim_{x\to 0} \frac{5x}{2}\right) \\
&= 1 \cdot 1 \cdot \frac{5}{2} \\
&= \frac{5}{2}.
\end{aligned}$$

(e) (8 points) $\lim_{x\to 0} \frac{\sqrt{5x+2}-x}{\sqrt{x+3}+2x}$

Solution: In this case, substituting $x = 0$ works fine:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{5x+2}-x}{\sqrt{x+3}+2x} &= \frac{\sqrt{0+2}-0}{\sqrt{0+3}+0} \\
&= \frac{\sqrt{2}}{\sqrt{3}}
\end{aligned}$$

5. (6 points) The following limit is known: $\lim_{x\to 0} \dfrac{1-\cos(x)}{\sin(3x)} = 0$. Explain why this is the case with explicit reference to the relevant known limits for trigonometric functions.
Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos(x)}{\sin(3x)} &= \lim_{x\to 0} \left(\frac{1-\cos(x)}{\sin(3x)}\right)\left(\frac{3x}{3x}\right) \\ &= \underbrace{\left(\lim_{x\to 0} \frac{1-\cos(x)}{x}\right)}_{=0} \underbrace{\left(\lim_{x\to 0} \frac{3x}{\sin(3x)}\right)}_{=1} \underbrace{\left(\lim_{x\to 0} \frac{1}{3}\right)}_{=\frac{1}{3}} \\ &= 0 \cdot 1 \cdot \frac{1}{3} = 0.\end{aligned}$$

6. (8 points) Electrical power P is related to the current I and voltage V by the equation $P = VI$.
   (a) (2 points) Solve the formula for the voltage V.
   Solution: $V = \dfrac{P}{I}$
   (b) (4 points) Using your answer to (a), if the power remains at a constant value $P = 2$, then what happens to voltage in the limit as $I \to 0^+$?
   Solution: $\lim_{I\to 0^+} V = \lim_{I\to 0^+} \dfrac{2}{I} = +\infty$
   (c) (2 points) What does this mean about reality? That is, please describe in your own words what part (b) suggests about how these quantities are related.
   Solution: If the power is held constant and the current goes to zero, then the voltage must increase toward infinity.

7. (14 points) Recall that a function is continuous at a point $x = a$ whenever $\lim_{x\to a} f(x) = f(a)$.
Find the value of k that makes the given piecewise function into a continuous function.
   (a) (7 points) $f(x) = \begin{cases} x^2 - 7x + 2, & x \geq k \\ x^2 + x + 1, & x < k. \end{cases}$
   Solution: We must find k so that

$$\lim_{x\to k^+} f(x) = \lim_{x\to k^-} f(x).$$

   So, compute the left-hand side:

$$\lim_{x\to k^+} f(x) = k^2 - 7k + 2$$

   and the right-hand side:

$$\lim_{x\to k^-} f(x) = k^2 + k + 1.$$

   In order for the function to be continuous at $x = k$, we need these two values to be equal, i.e.

$$k^2 - 7k + 2 = k^2 + k + 1.$$

   So, subtract k^2 to get $-7k + 2 = k + 1$ and thus $1 = 8k$, so $k = \dfrac{1}{8}$.
   (b) (7 points) $g(x) = \begin{cases} x^2 - 3x + 1, & x < 2 \\ 5x - k, & x \geq 2. \end{cases}$
   Solution: In this case, we need to find the value of k so that

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x).$$

   Similar to the previous problem, this results in the equation

$$4 - 6 + 1 = 10 - k,$$

   and so solving for k yields

$$k = 10 + 6 - 4 - 1 = 11.$$