Example of a limit

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What does it mean for

$$\lim_{n\to\infty} a_n = L$$

Def: $\forall\epsilon > 0 \exists N \in \mathbb{N} \forall n > N |a_n - L| < \epsilon$

How to prove $\lim_{n\to\infty} \underbrace{\frac{7n-17}{43n+2}}_{=:a_n} = \frac{7}{43}$?

Scratch work: Ultimately, we need to show

$$\forall\epsilon > 0 \exists N_\epsilon \forall n > N \left| a_n - \frac{7}{43} \right| < \epsilon$$

Consider our **ultimate goal**: $\left| \frac{7n-17}{43n+2} - \frac{7}{43} \right| < \epsilon$. By algebra,

$$\begin{aligned}
\left| \frac{7n-17}{43n+2} - \frac{7}{43} \right| &= \left| \frac{43(7n-17) - 301n - 14}{43(43n+2)} \right| \\
&= \left| \frac{301n - 731 - 301n - 14}{43(43n+2)} \right| \\
&= \left| \frac{-745}{43(43n+2)} \right| \\
&= \frac{745}{43(43n+2)} \overset{\text{need}}{<} \epsilon
\end{aligned}$$

Solve for n:

$$745 < 43(43n+2)\epsilon$$

$$\frac{745}{\epsilon} < (1849n + 86)$$

$$\frac{\frac{745}{\epsilon} - 86}{1849} < n$$

Proof: Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $N > \dfrac{\frac{745}{\epsilon} - 86}{1849}$. If $n > N$, then

$$\begin{aligned}
\left|a_n - \frac{7}{43}\right| &= \left|\frac{7n-17}{43n+2} - \frac{7}{43}\right| \\
&= \left|\frac{43(7n-17) - 301n - 14}{43(43n+2)}\right| \\
&= \left|\frac{301n - 731 - 301n - 14}{43(43n+2)}\right| \\
&= \left|\frac{-745}{43(43n+2)}\right| \\
&= \frac{745}{43(43n+2)}.
\end{aligned}$$

Since $n > N > \dfrac{\frac{745}{\epsilon} - 86}{1849}$. This means

$$43n > \frac{\frac{745}{\epsilon} - 86}{43}$$

$$43n + 2 > \frac{\frac{745}{\epsilon} - 86}{43} + 2 = \frac{\frac{745}{\epsilon}}{43}$$

$$43(43n+2) > \frac{745}{\epsilon}$$

$$\frac{43(43n+2)}{745} > \frac{1}{\epsilon}$$

Take reciprocal:

$$\frac{745}{43(43n+2)} < \epsilon,$$

completing the proof.