§3.4 – Impulses

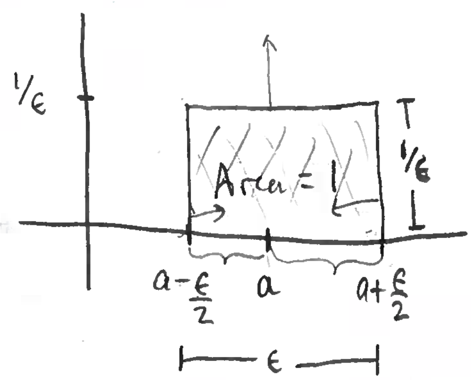
a large force occuring "in an instant"

↳ model hammer blows, medical injections, shock absorbers in cars

Dirac delta "function"

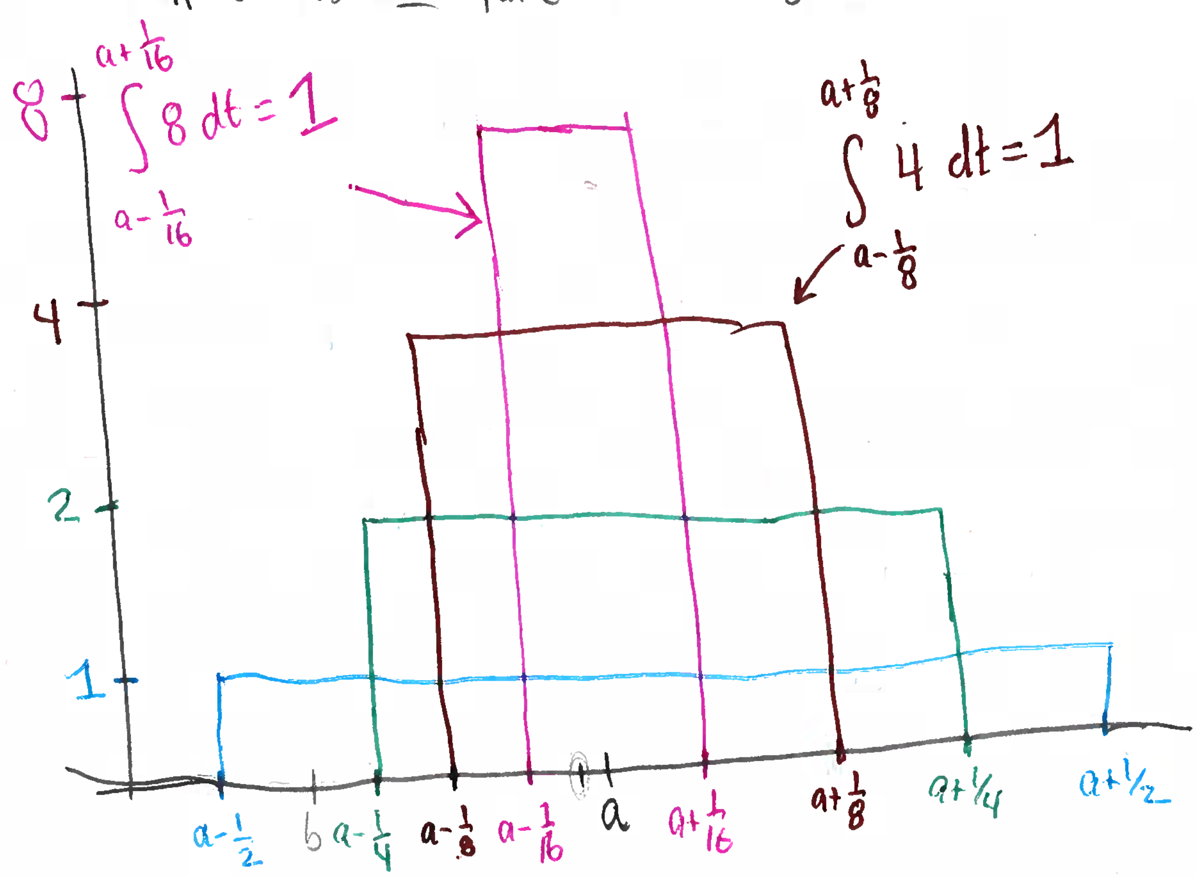
Consider an impulse over a short time period ϵ

small #



$$\int_{a-\epsilon/2}^{a+\epsilon/2} \frac{1}{\epsilon}\, dt = 1$$

What happens as I take $\epsilon \to 0$?



The "object" we obtain as $\epsilon \to 0$ is called the Dirac delta ~ $\delta_a(t)$

IT IS NOT A FUNCTION

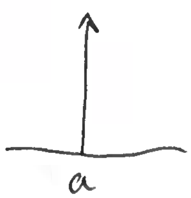
(but we pretend it is)

"it is a...
* distribution
* measure"

"Real" definition of $\delta_a(t)$:

("sifting property") (a>0)

$$\underbrace{\int_0^\infty \delta_a(t) f(t)\,dt}_{\text{integration against } \delta_a} = \underbrace{f(a)}_{\text{function evaluation at } a}$$



Consequence:

$$\mathcal{L}\{\delta_a(t)\}(s) \overset{def}{=} \int_0^\infty \delta_a(t)\, e^{-st}\, dt \overset{\text{sifting}}{=} e^{-sa}$$

$$\Downarrow$$

$$\mathcal{L}^{-1}\{e^{-sa}\} = \delta_a(t)$$

$\lambda^2+\lambda=0$

Ex : $\begin{cases} x''+x' = \delta_2(t) \\ x(0)=0,\ x'(0)=0 \end{cases}$

2 seconds in, hammer blow/electric impulse to system

$\Big\downarrow \mathcal{L}$

$$\left(s^2 X(s) - 0 - 0\right) + \left(s X(s) - 0\right) = e^{-2s}$$

$$X(s) = \frac{e^{-2s}}{s(s+1)}$$

$F(s) = \dfrac{1}{s(s+1)}$

$\Downarrow \quad s-(-1)$

$f(t) = \dfrac{1}{0-(-1)}\left(e^{0t} - e^{-t}\right)$

$= 1 - e^{-t}$

Therefore

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = H(t-2)\left(1 - e^{-(t-2)}\right)$$

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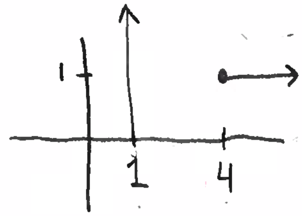
$$\begin{cases} x' + 3x = \underbrace{\delta_1(t) + H(t-4)} \\ x(0) = 1 \end{cases}$$



$$\Big\downarrow \mathcal{L}$$

$$\left(s\,X(s) - 1\right) + 3X(s) = e^{-s} + \frac{1}{s}e^{-4s}$$

$$X(s) = \frac{1 + e^{-s} + \frac{1}{s}e^{-4s}}{s+3}$$

$$= \frac{1}{s-(-3)} + \frac{1}{s-(-3)}e^{-s} + \frac{1}{s(s-(-3))}e^{-4s}$$

$$F(s) \Rightarrow f(t) = \frac{1}{0-(-3)}\left[1 - e^{-3t}\right] = \frac{1}{3}\left(1 - e^{-3t}\right)$$

$$\boxed{x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-3t} + H(t-1)e^{-3(t-1)} + H(t-4)\,\frac{1}{3}\left(1 - e^{-3(t-4)}\right)}$$