Notation: often $\mathcal{L}\{f\}(s)$ is written as

$F(s)$

capital F refers to Laplace xform of f

Theorem 3.2: for constants α, β and functions f, g,
(linear property)

$$\mathcal{L}\{\alpha f + \beta g\} = \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}$$

Ex: Let $f(t) = e^{at}$, $a \in \mathbb{R}$

$e^a e^b = e^{a+b}$

Compute

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{at} e^{-st}\, dt$$

$u = (a-s)t$

$\frac{1}{a-s} du = dt$

$$= \lim_{b\to\infty} \int_0^b e^{(a-s)t}\, dt$$

$$= \lim_{b\to\infty} \frac{1}{a-s} e^{(a-s)t} \Big|_{t=0}^{t=b}$$

$$= \lim_{b\to\infty} \left(\frac{1}{a-s}\right)\left[e^{(a-s)b} - 1\right]$$

limit $= 0$ only when
$a - s < 0$
$s > a$

$$= \frac{1}{a-s}\left[-1 + \lim_{b\to\infty} e^{(a-s)b}\right]$$

$$\overset{(s>a)}{=} \frac{1}{a-s}\left[-1 + 0\right]$$

$$= \frac{1}{s-a}$$

Ex: Let $g(t) = t$.

Compute

$$\mathcal{L}\{g\}(s) = \int_0^\infty t e^{-st}\,dt$$

probability

moment generating function

$u = t \quad dv = e^{-st}$

$du = dt \quad v = -\frac{1}{s}e^{-st}$

int by parts

$$= \lim_{b\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_{t=0}^{t=b} + \int_0^b \frac{1}{s}e^{-st}\,dt \right]$$

$$= \lim_{b\to\infty}\left[\left(-\frac{b}{s}e^{-sb} + 0\right) + \frac{1}{s}\left(-\frac{1}{s}\right)e^{-st}\Big|_0^b \right]$$

$$= \lim_{b\to\infty}\left[\cancel{\frac{-b}{se^{sb}}}^{\,0} - \frac{1}{s^2}\left(\cancel{e^{-sb}}^{\,0} - 1\right) \right]$$

go to 0 iff $s > 0$

need $-sb < 0$

$sb > 0$

$s > 0$

L'Hôpital $\frac{\infty}{\infty}$

$$\lim_{b\to\infty} \frac{b}{e^{sb}} = \lim_{b\to\infty} \frac{1}{se^{sb}} \underset{(s>0)}{=} 0$$

$$\underset{(s>0)}{=} \frac{1}{s^2}$$

Inverse Laplace transform

$$\text{diffeq} \xrightarrow{\mathcal{L}} \text{algebra} \xrightarrow{\mathcal{L}^{-1}} \text{solution}$$

$$\mathcal{L}\{e^{ta}\} = \frac{1}{s-a}$$

transform domain

$$\Updownarrow$$

$$e^{ta} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

"time domain"

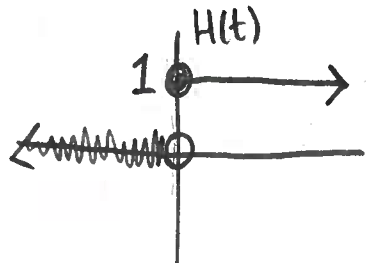
$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

Ex 3.7 Heaviside function ("switch" function)

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

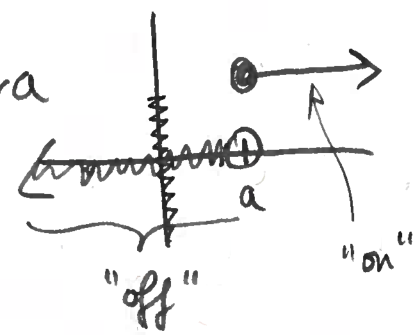


$t-a \geq 0$
$t \geq a$

$t-a < 0$
$t < a$

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

$$\int_a^b = \int_a^c + \int_c^b$$



Compute

$(a \geq 0)$

$$\mathcal{L}\{H(t-a)\}(s) = \int_0^\infty H(t-a)e^{-st}\,dt$$

$$= \int_0^a H(t-a)e^{-st}\,dt + \int_a^\infty H(t-a)e^{-st}\,dt$$

$0<t<a$

$$= 0 + \lim_{b\to\infty}\int_a^b 1\cdot e^{-st}\,dt$$

$$= \lim_{b\to\infty}\left(-\frac{1}{s}\right)e^{-st}\Big|_a^b$$

$$= \lim_{b\to\infty}\left(\cancelto{0}{\frac{-e^{-sb}}{s}} + \frac{e^{-sa}}{s}\right)$$

$(s>0)$

$$= \frac{1}{s}e^{-sa}$$

Ex 3.8: $f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 4, & 2 \leq t \leq 3 \\ 2, & 3 < t \leq 6 \\ 0, & t > 6 \end{cases}$

"3" on starting at 0

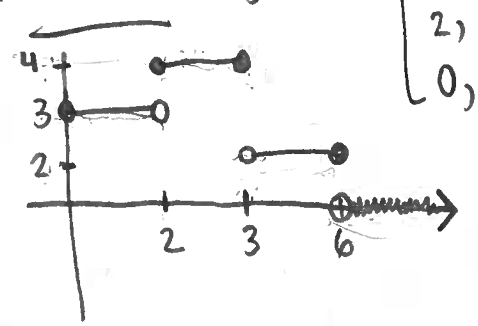
at 2...turn on 4, turn off 3

$$= 3H(t) + (4-3)H(t-2)$$

turn 2 off

$$+ (2-4)H(t-3) - 2H(t-6)$$

2 "on"   4 "off"   at 3



$$= 3H(t) + H(t-2) - 2H(t-3) - 2H(t-6)$$

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{3H(t) + H(t-2) - 2H(t-3) - 2H(t-6)\}$$

$$= 3\mathcal{L}\{H(t)\} + \mathcal{L}\{H(t-2)\} - 2\mathcal{L}\{H(t-3)\} - 2\mathcal{L}\{H(t-6)\}$$

$$= \frac{3}{s} + \frac{1}{s}e^{-2s} - \frac{2}{s}e^{-3s} - \frac{2}{s}e^{-6s}$$

Shift Property

$$\mathcal{L}\{f(t)e^{at}\}(s) = \underbrace{F(s-a)}_{\mathcal{L}\{f\}(s-a)}$$

Switching Property

$$\mathcal{L}\{H(t-a)f(t-a)\} = \underbrace{e^{-as}F(s)}_{e^{-as}\mathcal{L}\{f\}(s)}$$