Ex: Solve $_0F_1$ hypergeometric diff-eq

$$tx'' + bx' - x = 0$$

b ≠ negative integer

Soln: $x(t) = \sum_{k=0}^{\infty} a_k t^k$

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1} \qquad x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2}$$

$$bx' = b\sum_{k=1}^{\infty} k a_k t^{k-1} \qquad tx''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-1}$$

add one to k

$$= b\sum_{k=0}^{\infty} (k+1) a_{k+1} t^k \qquad = \sum_{k=1}^{\infty} (k+1)(k) a_{k+1} t^k$$

Plug into DE:

$$\sum_{k=1}^{\infty} (k+1)k\, a_{k+1} t^k + b\sum_{k=0}^{\infty} (k+1) a_{k+1} t^k - \sum_{k=0}^{\infty} a_k t^k = 0$$

$$ba_1 + \sum_{k=1} \sim \qquad -a_0 - \sum_{k=1} \sim$$

$$(ba_1 - a_0) + \sum_{k=1}^{\infty} \left[(k+1)k\, a_{k+1} + b(k+1) a_{k+1} - a_k \right] t^k = 0 + 0t + 0t^2 + 0t^3 + \ldots$$

$$\begin{cases} ba_1 - a_0 = 0 \longrightarrow \boxed{a_1 = \dfrac{a_0}{b}} \\ [(k+1)k + b(k+1)]\, a_{k+1} - a_k = 0 \longrightarrow \boxed{a_{k+1} = \dfrac{a_k}{(k+1)(k+b)}} \longleftarrow k = 1, 2, 3, \ldots \end{cases}$$

②

"Pochhammer symbol"

$(b)_4 = b(b+1)(b+2)(b+3)$

"rising factorial" $b^{\overline{4}}$

$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)$

k	$a_{k+1} = \frac{a_k}{(k+1)(k+b)}$ ← $k = 1, 2, 3, \ldots$
0	$a_1 = \frac{a_0}{b}$
1	$a_2 = \frac{a_1}{2\cdot(1+b)} = \frac{a_0}{2\cdot b(1+b)}$
2	$a_3 = \frac{a_2}{3\cdot(2+b)} = \frac{a_0}{(2\cdot 3)\, b(1+b)(2+b)}$
3	$a_4 = \frac{a_3}{4\cdot(3+b)} = \frac{a_0}{4!\, b(1+b)(2+b)(3+b)} = \frac{a_0}{4!\,(b)_4}$
4	$a_5 = \frac{a_4}{5\cdot(4+b)} = \frac{a_0}{5!\,(b)_5}$

$\vdots$

$$x(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$

$$= a_0 + \frac{a_0}{b} t + \frac{a_0}{2!\,(b)_2} t^2 + \frac{a_0}{3!\,(b)_3} t^3 + \ldots$$

$$= a_0 \sum_{k=0}^{\infty} \frac{t^k}{k!\,(b)_k} =: {}_0F_1(\,;b;t)$$

In general:

$${}_pF_q(a_1,\ldots,a_p; b_1,\ldots,b_q; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{t^k}{k!}$$

Ex: $_2F_1$ ✓

$$\overbrace{(1-t)t x''}^{tx''-t^2x''} + \overbrace{(c-(a+b+1)t)x'}^{cx'-(a+b+1)tx'} - abx = 0$$

②

$$x = \sum_{k=0}^{\infty} a_k t^k \qquad x' = \sum_{k=1}^{\infty} k a_k t^{k-1} \qquad x'' = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-2}$$

5th minus $\longrightarrow abx = ab\sum_{k=0}^{\infty} a_k t^k = aba_0 + aba_1 t + \sum_{k=2} \sim$

1st $\longrightarrow tx'' = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-1} = \sum_{k=1}^{\infty} (k+1)k\,a_{k+1} t^k = 2a_2 t + \sum_{k=2} \sim$

2nd (minus) $\longrightarrow t^2 x'' = \sum_{k=2}^{\infty} k(k-1)a_k t^k$

3rd $\longrightarrow cx' = c\sum_{k=1}^{\infty} k a_k t^{k-1} = c\sum_{k=0}^{\infty}(k+1)a_{k+1} t^k = ca_1 + c\cdot 2a_2 t + \sum_{2}^{\infty} \sim$

4th (minus) $\longrightarrow (a+b+1)tx' = (a+b+1)\sum_{k=1}^{\infty} k a_k t^k = (a+b+1)a_1 t + \sum_{2} \sim$

~~[c + aba_0] a_1~~

$$[ca_1 - aba_0] + [2a_2 + 2a_2 c - (a+b+1)a_1 - aba_1]t + \sum_{k=2}^{\infty}\left[\begin{matrix}\overset{1st}{(k+1)k\,a_{k+1}} - \overset{2nd}{k(k-1)a_k} \\ + \underset{3rd}{c(k+1)a_{k+1}} - \underset{4th}{(a+b+1)k a_k} \\ \underset{5th}{-aba_k}\end{matrix}\right] t^k = 0$$

4

$$\begin{cases} ca_1 - aba_0 = 0 \\ 2a_2 + 2a_2 c - (a+b+1)a_1 - aba_1 = 0 \\ \left[k(k+1) + c(k+1)\right] a_{k+1} + \left[-k(k-1) - (a+b+1)k - ab\right] a_k = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} a_1 = \frac{ab}{c} a_0 \\ a_2 = \frac{\left[a+b+1+ab\right] a_1}{2(c+1)} = \frac{(a+b+1+ab)ab}{2(c+1)} a_0 \\ a_{k+1} = \frac{\left(k(k-1) + (a+b+1)k + ab\right) a_k}{k(k+1) + c(k+1)} \end{cases}$$

$$\Downarrow$$

$$k=2 \rightarrow a_3 = \frac{\left(2(1) + (a+b+1)2 + ab\right) a_2}{\underbrace{2(3) + 3c}_{3(2+c)}} = \frac{(2+2(a+b+1)+ab)(a+b+1+ab)ab a_0}{2\cdot 3(2+c)(1+c)}$$

TURNS OUT ...

$$x(t) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{t^k}{k!} =: {}_2F_1(a,b;c;t)$$