Ex: Solve Airy equation

find 1st four nonzero terms of soln to

$$x'' - tx = 0$$

Soln: "Guess" $x(t) = \sum_{k=0}^{\infty} a_k t^k$

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}$$

$$x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2}$$

$2a_2 t^0 + 2 \cdot 3 a_3 t^1$

$$t x(t) = \sum_{k=0}^{\infty} a_k t^{k+1}$$

$a_0 t + a_1 t^2 + \dots$

$k \mapsto k-1$

$$= \sum_{k=1}^{\infty} a_{k-1} t^k$$

$$x''(t) = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k = 2a_2 + 3 \cdot 2 a_3 t + \dots$$

So:

$$0 = x'' - tx = \underbrace{\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k} - \sum_{k=1}^{\infty} a_{k-1} t^k$$

$$= 2a_2 + \sum_{k=1}^{\infty} \underbrace{\left[(k+2)(k+1) a_{k+2} - a_{k-1} \right]} t^k$$

need $= 0$

need $= 0$

recurrence relation

$$\Downarrow$$

$$2a_2 = 0 \rightarrow \boxed{a_2 = 0}$$

Solve recurrence relation for a_{k+2}:

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)} \quad \text{for } k = 1, 2, 3, \ldots$$

k	a_{k+2}
0	$a_2 = 0$ ← from last page... not technically a k we can use
1	$a_3 = \frac{a_0}{3 \cdot 2}$
2	$a_4 = \frac{a_1}{4 \cdot 3}$
3	$a_5 = \frac{a_2}{5 \cdot 4} = 0$
4	$a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{(3 \cdot 6)(2 \cdot 5)}$
5	$a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{(7 \cdot 4)(6 \cdot 3)}$
6	$a_8 = \frac{a_5}{8 \cdot 7} = 0$
⋮	

We get

$$x(t) = \sum_{k=0}^{\infty} a_k t^k = a_0 + a_1 t + 0t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + a_7 t^7 + 0t^8 + \ldots$$

$$= a_0 + a_1 t + \frac{a_0}{3 \cdot 2} t^3 + \frac{a_1}{4 \cdot 3} t^4 + \frac{a_0}{(3 \cdot 6)(2 \cdot 5)} t^6 + \frac{a_1}{(7 \cdot 4)(6 \cdot 3)} t^7 + \ldots$$

$$= a_0 \left(1 + \frac{1}{3 \cdot 2} t^2 + \frac{1}{(3 \cdot 6)(2 \cdot 5)} t^6 + \ldots \right)$$

$$+ a_1 \left(t + \frac{1}{4 \cdot 3} t^3 + \frac{1}{(7 \cdot 4)(6 \cdot 3)} t^7 + \ldots \right)$$

Bessel 1^{st} kind, order $n=1$

$$t^2x'' + tx' + (t^2-1)x = 0$$

Guess: $x(t) = \sum_{k=0}^{\infty} a_k t^k$

$$\rightarrow x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}$$

$$x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2}$$

$$\rightarrow \checkmark\ t^2 x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^k$$

$$\rightarrow \checkmark\ t x'(t) = \sum_{k=1}^{\infty} k a_k t^k$$

$$\rightarrow t^2 x(t) = \underbrace{\sum_{k=0}^{\infty} a_k t^{k+2}} = \sum_{k=2} a_{k-2} t^k$$

Plug into the DE:

$$\sum_{k=2}^{\infty} k(k-1) a_k t^k + \sum_{k=1}^{\infty} k a_k t^k + \sum_{k=2}^{\infty} a_{k-2} t^k - \sum_{k=0}^{\infty} a_k t^k = 0$$

$$\cancel{a_1 t} \underbrace{- a_0}_{=0} \cancel{- a_1 t} + \sum_{k=2}^{\infty} \underbrace{\left[k(k-1) a_k + k a_k + a_{k-2} - a_k \right]}_{=0} t^k = 0$$

$$k(k-1)a_k + k a_k + a_{k-2} - a_k = 0$$

$$\underbrace{\left[k(k-1)+k-1\right]}\, a_k = -a_{k-2}$$

$$a_k = \frac{-a_{k-2}}{k(k-1)+k-1}, \quad k=2,3,4,\ldots$$

k	a_k
0	$a_0 = 0$
2	$a_2 = \frac{-a_0}{} = 0$
3	$a_3 = \frac{-a_1}{3(2)+3-1} = \frac{-a_1}{8}$
4	$a_4 = \frac{-a_2}{} = 0$
5	$a_5 = \frac{-a_3}{5(4)+5-1} = \frac{-a_3}{24} = \frac{a_1}{192}$
6	$a_6 = 0$
7	$a_7 = \frac{-a_5}{7\cdot 6+7-1} = \frac{-a_5}{48} = \frac{-a_1}{9216}$
8	$a_8 = 0$
9	$a_9 = \frac{-a_7}{9\cdot 8+9-1} = \frac{-a_7}{80} = \frac{a_1}{737280}$

Solution is

$$x(t) = \sum_{k=0}^{\infty} a_k t^k = a_1 t + \left(-\frac{a_1}{8} t^3\right) + \frac{a_1}{192} t^5 - \frac{a_1}{9216} t^7 + \frac{a_1}{737280} t^9 - \ldots$$

$$= a_1 \left[t - \frac{1}{8} t^3 + \frac{1}{192} t^5 - \frac{1}{9216} t^7 + \frac{1}{737280} t^9 - \ldots \right]$$