EX: Use series method to solve

$$x' = x$$

we know soln is

$$x(t) = Ce^t$$

$$x' - x = 0$$

Recall

$a_k = \frac{1}{k!}$

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{t^2}{2!} + \ldots$$

Power series

$$\sum_{k=0}^{\infty} a_k t^k$$

some constant depending on k

Soln: Make guess

$$x(t) = \sum_{k=0}^{\infty} a_k t^k$$

$$= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$$

$$x'(t) = \sum_{k=1}^{\infty} a_k k t^{k-1}$$

$\frac{4}{2} = 2$

$$= a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \ldots$$

$$= \sum_{k=0}^{\infty} (a_{k+1})(k+1) t^k$$

Since DE is $x' = x$,

$$\sum_{k=0}^{\infty} (k+1) a_{k+1} t^k = \sum_{k=0}^{\infty} a_k t^k$$

$$\sum_{k=0}^{\infty} \underbrace{[(k+1)a_{k+1} - a_k]}_{\text{must have these coeffs } = 0 \text{ for all } t} t^k = 0$$

So consider

$$(k+1)a_{k+1} - a_k = 0 \qquad k = 0, 1, 2, 3, \ldots$$

$$a_{k+1} = \frac{a_k}{k+1}$$

$4! = 4 \cdot 3 \cdot 2 \cdot 1$

$k =$	eq^t
0	$a_1 = \frac{a_0}{1!} = a_0$
1	$a_2 = \frac{a_1}{2} = \frac{a_0}{2!}$
2	$a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3} = \frac{a_0}{3!}$
3	$a_4 = \frac{a_3}{4} = \frac{a_0}{4 \cdot 3!} = \frac{a_0}{4!}$
⋮	
N	$a_{N+1} = \frac{a_0}{(N+1)!}$

So going back to

$$x(t) = \sum_{k=0}^{\infty} a_k t^k$$

$$= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots$$

$$= a_0 + a_0 t + \frac{a_0}{2!} t^2 + \frac{a_0}{3!} t^3 + \ldots$$

$$= a_0 \left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots \right]$$

$$= a_0 e^t$$

Enforce $x(0) = 5$: $x(t) = a_0 + 0 + 0 + 0 + \ldots = 5 \Rightarrow a_0 = 5$

Ex: Solve $x'' + x = 0$

$$\cos(t) = \sum_{k=0}^{\infty} \frac{x^{2k}(-1)^k}{(2k)!}$$

$$\sin(t) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\boxed{\begin{gathered}\lambda^2 + 1 = 0 \to \lambda = \pm i \\ x(t) = c_0 \cos(t) + c_1 \sin(t)\end{gathered}}$$

Guess $x(t) = \sum_{k=0}^{\infty} a_k t^k$

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}$$

$$x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} t^k$$

$$0 = x''(t) + x(t) = \sum_{k=0}^{\infty} \underbrace{\left[(k+2)(k+1)a_{k+2} + a_k\right]}_{\text{need} = 0 \text{ for all } k} t^k$$

$$\Rightarrow \quad a_{k+2} = \frac{-a_k}{(k+2)(k+1)}, \quad k = 0, 1, 2, \ldots$$

$k=$	recurrence
0	$a_2 = \frac{-a_0}{2\cdot 1} = \frac{-a_0}{2!}$
1	$a_3 = \frac{-a_1}{3\cdot 2} = \frac{-a_1}{3!}$
2	$a_4 = \frac{-a_2}{4\cdot 3} = \frac{a_0}{4!}$
3	$a_5 = \frac{-a_3}{5\cdot 4} = \frac{a_1}{5!}$
4	$a_6 = \frac{-a_4}{6\cdot 5} = \frac{-a_0}{6!}$
5	$a_7 = \frac{-a_5}{7\cdot 6} = \frac{-a_1}{7!}$

$\vdots$

So we have

$$x(t) = \sum_{k=0}^{\infty} a_k t^k = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots$$

$$= \underline{a_0} + a_1 t - \underbrace{\frac{a_0}{2!}}\, t^2 - \frac{a_1}{3!} + \underbrace{\frac{a_0}{4!}}\, t^4 + \frac{a_1}{5!} t^5 - \cdots$$

$$= a_0\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right] + a_1\left[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right]$$

$$= a_0 \cos(t) + a_1 \sin(t)$$