Ex 2.18 : $x'' - x' + 7x = 5t - 3$

Step 1 : Solve

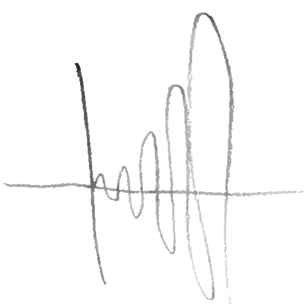
$$x'' - x' + 7x = 0$$

$$x(t) = e^{\lambda t} \rightarrow \lambda^2 - \lambda + 7 = 0$$

$$\downarrow$$

$$\lambda = \frac{1 \pm \sqrt{1-4(7)}}{2}$$

$$= \frac{1 \pm \sqrt{27}\, i}{2} = \frac{1 \pm 3\sqrt{3}\, i}{2} = \alpha \pm \beta i$$



homogen soln :

$$x_h(t) = c_1 e^{t/2} \cos\left(\frac{3\sqrt{3}}{2}t\right) + c_2 e^{t/2} \sin\left(\frac{3\sqrt{3}}{2}t\right)$$

Step 2 : Guess : $x_p(t) = A_0 + A_1 t$

find

Plug x_p into nonhomog ODE :

$$x_p' = A_1 \qquad x_p'' = 0$$

$$0 - A_1 + 7(A_0 + A_1 t) = 5t - 3$$

$$(7A_0 - A_1) + 7A_1 t = 5t - 3$$

$$7A_1 = 5 \rightarrow A_1 = \frac{5}{7}$$

$$7A_0 - A_1 = -3$$

$$7A_0 - \frac{5}{7} = -3 = \frac{-21}{7} \rightarrow 7A_0 = -\frac{16}{7} \rightarrow A_0 = \frac{-16}{49}$$

$$x(t) = c_1 e^{t/2} \cos\left(\frac{3\sqrt{3}}{2}t\right) + c_2 e^{t/2} \sin\left(\frac{3\sqrt{3}}{2}t\right) + \left(-\frac{16}{49} + \frac{5}{7}t\right)$$

Ex 2.19: $x'' - x = 5e^{-t}$

Step 1: Solve $x'' - x = 0 \sim x(t) = e^{\lambda t}$

$$\lambda^2 - 1 = 0 \to \lambda = \pm 1$$

$$\Rightarrow x(t) = c_1 e^{t} + c_2 e^{-t}$$

Step 2: [Chart (naively) suggests $x_p(t) = Ae^{-t}$

$\Downarrow$

(you can check)

$x_p'' - x = 0$ fails

$Ae^{-t} - Ae^{-t} = 0$ ✓ $\longrightarrow$ this shows that particular x_p does not solve nonhomog eqt]

Using Remark 2.15:

$$\rightarrow x_p(t) = Ate^{-t}$$

$$x_p'(t) = Ae^{-t} - Ate^{-t}$$

$$x_p''(t) = -Ae^{-t} - A\left[e^{-t} - te^{-t}\right] = -2Ae^{-t} + Ate^{-t}$$

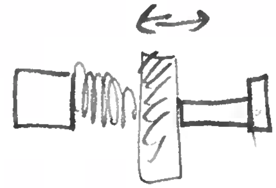
$$x_p'' - x_p = \left[-2Ae^{-t} + \cancel{Ate^{-t}}\right] - \left[\cancel{Ate^{-t}}\right] \overset{\text{set}}{=} 5e^{-t}$$

$$-2A = 5 \rightarrow \boxed{A = -\frac{5}{2}}$$

Therefore gen soln is

$$x(t) = c_1 e^{t} + c_2 e^{-t} - \frac{5}{2} te^{-t}$$

Ex 2.21 $x'' + 2x = \sin(3t)$



Step 1: $\lambda^2 + 2 = 0$

$$\lambda = \pm\sqrt{2}\, i$$

$$\rightarrow x_h(t) = c_1\cos(\sqrt{2}\, t) + c_2\sin(\sqrt{2}\, t)$$

Step 2: $x_p(t) = A\sin(3t)$

$$x_p' = 3A\cos(3t)$$

$$x_p'' = -9A\sin(3t)$$

$$-9A\sin(3t) + 2A\sin(3t) = \sin(3t)$$

$(-7A-1)\sin(t) = 0$ $-7A\sin(3t) = \sin(3t)$

$$-7A = 1 \rightarrow \boxed{A = -\frac{1}{7}}$$

Gen soln:

$$x(t) = c_1\cos(\sqrt{2}\, t) + c_2\sin(\sqrt{2}\, t) - \frac{1}{7}\sin(3t)$$