§2.2 $\underbrace{\text{2nd order}}_{\text{2nd derivative}}$ $\underbrace{\text{linear}}_{\text{see form}}$ $\underbrace{\text{homogeneous}}_{\text{no term depending only on } t}$ Diff Eqs w/ constant coefficients.

form:

homogeneous

2nd order

$$ax'' + bx' + cx = 0$$

a, b, c constants

Linear Algebra

$$T(f) = a\frac{d^2f}{dt^2} + b\frac{df}{dt} + cf$$

$$\ker T = \{ f \in V : T(f) = 0 \}$$

functions

Earlier: with 1st order ODE's ~ initial condition of form $x(0) = x_0$

Now: initial conditions

$$\begin{cases} x(0) = x_0 \\ x'(0) = x_1 \end{cases}$$

Theorem 2.3: The IVP $\begin{cases} ax'' + bx' + cx = 0 \\ x(0) = x_0,\ x'(0) = x_1 \end{cases}$

has unique soln that exists on $\mathbb{R}$.

FACT: There will be exactly two "independent" solutions to these DE's ~ $x_1(t)$, $x_2(t)$

not a constant multiple of each other

And general soln will look like

Thm 2.4: $x(t) = c_1 x_1(t) + c_2 x_2(t)$

How to solve $ax'' + bx' + cx = 0$?

"Guess" solution of form: $\boxed{x(t) = e^{\lambda t}}$ ← unknown constant
("ansatz")

$$x'(t) = \lambda e^{\lambda t}$$
$$x''(t) = \lambda^2 e^{\lambda t}$$

Plug into DE:

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t} = 0$$

nonzero

⇓

divide by $e^{\lambda t}$

⇓

$$a\lambda^2 + b\lambda + c = 0 \leftarrow$$ characteristic equation for the DE

know

Lin Algebra

e-vals of matrix

$$\underset{2\times 2}{M}\underset{2\times 1}{v} = \underset{\text{e-value}}{\lambda}\underset{2\times 1}{v}$$ ← e-vector

⇓ QF

eigenvalues → $$\boxed{\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

"eigen" ~ German means "your"

Case 1: $b^2 - 4ac > 0$

$\Rightarrow$ get two distinct real e-values

$$\lambda_1 = \frac{-b + \sqrt{b^2-4ac}}{2a}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2-4ac}}{2a}$$

$\Rightarrow$ general soln

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Case 2: $b^2 - 4ac = 0$

$\Rightarrow$ one e-value $\lambda = \frac{-b}{2a}$ $\Rightarrow$ $x_1(t) = e^{\lambda t}$ is a soln

Turns out $x_2(t) = te^{\lambda t}$ will be a soln.

will see later

$\Rightarrow$ general soln

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

Case 3: $b^2 - 4ac < 0$

$i = \sqrt{-1}$

$\sqrt{-4} = \sqrt{4}\, i = 2i$

$\Rightarrow$ $\sqrt{b^2-4ac} = \sqrt{4ac-b^2}\, i$

$\Rightarrow$ evalues $\lambda_1 = \frac{-b + \sqrt{4ac-b^2}\, i}{2a}$ $=: \alpha + \beta i$

$\lambda_2 = \frac{-b - \sqrt{4ac-b^2}\, i}{2a}$ $=: \alpha - \beta i$

$\Rightarrow$ general soln

$$x(t) = c_1 e^{(\alpha+\beta i)t} + c_2 e^{(\alpha-\beta i)t}$$

Ex: Consider

$$x'' - x' - 12x = 0$$

Guess $x = e^{\lambda t} \to x' = \lambda e^{\lambda t} \to x'' = \lambda^2 e^{\lambda t}$

$$\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 12 e^{\lambda t} = 0$$

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0$$

$$\lambda = 4, -3$$

Therefore,

$$x(t) = c_1 e^{4t} + c_2 e^{-3t}$$

Consider initial conditions $x(0) = 1$, $x'(0) = 2$.

$$x'(t) = 4c_1 e^{4t} - 3c_2 e^{-3t}$$

$$\underbrace{1}_{\text{given}} = x(0) = \underbrace{c_1 + c_2}_{\text{computed}} \quad (i)$$

$$\underbrace{2}_{\text{given}} = x'(0) = \underbrace{4c_1 - 3c_2}_{\text{computed}} \quad (ii)$$

$(i) \to \boxed{c_1 = 1 - c_2}$

plug into (ii)

$$2 = 4(1 - c_2) - 3c_2$$
$$= 4 - 4c_2 - 3c_2$$
$$-2 = -7c_2 \to \boxed{c_2 = \frac{2}{7}}$$

$$c_1 = 1 - \frac{2}{7} = \frac{5}{7}$$

particular soln is

$$\Longrightarrow \quad x(t) = \frac{5}{7} e^{4t} + \frac{2}{7} e^{-3t}$$