First order linear

$$x^{(1)} + p(t)x = q(t)$$

How to solve?

* mult by int. factor $\mu(t) = e^{\int p}$

Structure Theorem: Solution to $x' + p(t)x = q(t)$ is
(Thm 1.233) of the form

$$x(t) = \underbrace{x_h(t)}_{\substack{\text{Soln of}\\ \text{homogeneous}\\ \text{eqt}\\ x'+p(t)x=0}} + \underbrace{x_p(t)}_{\substack{\text{ANY solution of}\\ \text{nonhomogeneous}}}$$

Ex (1.24):

$$x' - 3x = e^{-t}$$

$$x' + \underbrace{(-3)}_{p(t)=-3}x = \underbrace{e^{-t}}_{q(t)=e^{-t}}$$

Let $\mu(t) = e^{\int p(t)dt} = e^{\int -3dt} = e^{-3t}$

$$e^{-3t}x' + (-3)e^{-3t}x = e^{-3t}e^{-t}$$

$$\frac{d}{dt}\left(xe^{-3t}\right) = e^{-4t}$$

$\downarrow \int$

$$xe^{-3t} = \int e^{-4t}dt = -\frac{1}{4}e^{-4t} + C$$

$$x(t) = e^{3t}\left[-\frac{1}{4}e^{-4t} + C\right]$$

$$= \underbrace{-\frac{1}{4}e^{-t}}_{x_p(t)} + \underbrace{Ce^{3t}}_{x_h(t)}$$

$x' + px = q$

$a\ln(b) = \ln(b^a)$

p.41 #2a

$x' = -\left(\frac{2}{t}\right)x + t$

$x' + \underbrace{\left(\frac{2}{t}\right)}_{p(t)}x = \underbrace{t}_{q(t)}$

$\mu(t) = e^{2\int \frac{1}{t}dt} = e^{2\ln(t)} = \cancel{e}^{\cancel{\ln}(t^2)} = t^2$

mult by $\mu(t)$

$t^2x' + 2tx = t^3$

$\frac{d}{dt}(x(t)t^2) = t^3$

$\downarrow \int$

$xt^2 = \frac{t^4}{4} + C$

$\boxed{x(t) = \frac{t^2}{4} + \frac{C}{t^2}}$

$x' + p(t)x = q(t)$

$\mu = e^{\int p}$

$= \exp(\int p)$

#2e)

$\theta' = -a\theta + \underbrace{\exp(bt)}_{}$ e^{bt}

$\theta' + a\theta = e^{bt}$

$\mu(t) = e^{\int a\,dt} = e^{at}$

$e^{at}\theta' + ae^{at}\theta = e^{at}e^{bt}$

$\frac{d}{dt}(\theta e^{at}) = e^{(a+b)t}$

$\downarrow \int$

$\theta e^{at} = \int e^{(a+b)t}\,dt = \frac{1}{a+b}e^{(a+b)t} + C$

$\theta(t) = \frac{1}{a+b}e^{bt} + Ce^{-at}$

$x' + px = q$

#3f $\begin{cases} R' = \frac{R}{t} + te^{-t} \\ R(1) = 1 \end{cases}$

$$R' - \frac{1}{t}R = \underbrace{te^{-t}}_{q(t)}$$

$p(t) = -\frac{1}{t}$

$$\mu = e^{\int p} = e^{-\ln(t)} = \frac{1}{e^{\ln t}} = \frac{1}{t}$$

$$\frac{1}{t}R' - \frac{1}{t^2}R = e^{-t}$$

$$\frac{d}{dt}\left(\frac{1}{t}R\right) = e^{-t}$$

$\downarrow \int$

$$\frac{1}{t}R = \int e^{-t}dt = -e^{-t} + C$$

$$\boxed{R(t) = -te^{-t} + tC}$$

$$\underbrace{1}_{\text{given}} = R(1) = \underbrace{-e^{-1}}_{-\frac{1}{e}} + C \rightarrow \boxed{C = \frac{1}{e} + 1}$$

$$R(t) = -te^{-t} + \left(1 + \frac{1}{e}\right)t$$

§1.4.2 Applications

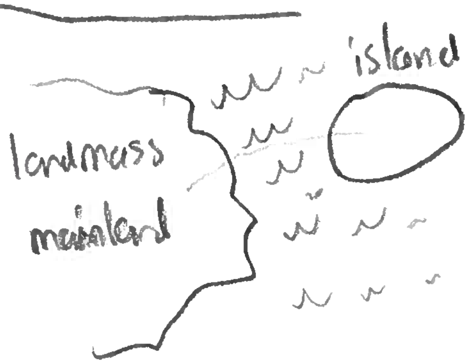
Ex 1.26 (Improved Newton's law of cooling)

Assume environment temp changes

$$T' = -h(T - \underbrace{Q(t)}_{\text{temp in environment at time } t}), \quad T(t_0) = T_0$$

$$T(t) = Ce^{-ht} + e^{-ht}\int_{t_0}^{t} hQ(s)e^{hs}\,ds$$

Exercise 8 p. 47



P ~ constant, # of species on mainland

$S(t)$ ~ # species on island at time t

$$S' = \chi - \mu$$

χ: colonization rate

μ: extinction rate (not int factor)

$$S' = I\left(1 - \frac{S}{P}\right) - \frac{E}{P}S$$

I: max col. rate

E: max extinction rate

a) equilibrium: means set $S' = 0$

$$0 = I\left(1 - \frac{S}{P}\right) - \frac{E}{P}S$$

$$-I = -\frac{IS}{P} - \frac{ES}{P}$$

$$S\left(\frac{I}{P} + \frac{E}{P}\right) = I$$

testable claim!! $\rightarrow$ $\boxed{S = \frac{IP}{I+E}}$