Ex: Solve

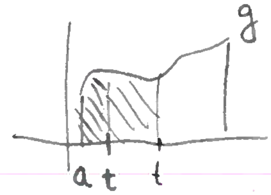
$$\begin{cases} x' = e^{-t^2} \\ x(0) = 2 \end{cases}$$

$$\int t e^{-t^2} dt$$

$$x = \underbrace{\int e^{-t^2} dt}_{\text{can't express this antiderivative in terms of common functions}} + C$$

Thm 1.12: If g is continuous, then for any $a \in \mathbb{R}$

$$\frac{d}{dt} \int_a^t g(s)\, ds = g(t).$$



Ex: It is common to express functions as an integral w/ variable limit

$$\ln(t) = \int_1^t \frac{1}{s} ds, \quad t > 0$$

optics
civil eng. → Fresnel integral: $C(t) = \int_0^t \cos(s^2) ds$

signal processing → Sine integral fnct: $Si(t) = \int_0^t \frac{\sin(s)}{s} ds$

Ex: $\begin{cases} x' = e^{-t^2} \\ x(0) = 2 \end{cases} \longrightarrow \int_0^t x'(s)\,ds = \int_0^t e^{-s^2}\,ds$

$$x(t) - x(0) = \int_0^t e^{-s^2}\,ds$$

$$x(t) = x(0) + \int_0^t e^{-s^2}\,ds$$

$$= 2 + \int_0^t e^{-s^2}\,ds$$

(note: $\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2}\,ds$)

$$\frac{\sqrt{\pi}}{2} \operatorname{erf}(t) = \int_0^t e^{-s^2}\,ds \to = 2 + \frac{\sqrt{\pi}}{2} \operatorname{erf}(t)$$

Exercises

(#1) $\begin{cases} x' = t\cos(t^2) \checkmark \\ x(0) = 1 \end{cases}$

$$x = \int t\cos(t^2)\,dt = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C$$

$u = t^2$ $\qquad = \frac{1}{2}\sin(t^2) + C$

$\frac{1}{2}du = t\,dt$

$$\underset{\substack{\uparrow \\ \text{given}}}{1} = x(0) = \underbrace{\frac{1}{2}\sin(0^2) + C}_{\text{computed}}$$

$1 = C \Rightarrow$ "particular soln" is

$$x(t) = \frac{1}{2}\sin(t^2) + 1$$

Ex #3 $\begin{cases} x'' = -3\sqrt{t} \checkmark \\ x(1) = 1,\ x'(1) = 2 \end{cases}$

$$x' = -3\int t^{1/2}\,dt = \frac{-3t^{3/2}}{3/2} + C$$

$$= -2t^{3/2} + C$$

$\downarrow \int$

$$x = \int -2t^{3/2} + C\,dt$$

$$= \frac{-2t^{5/2}}{5/2} + Ct + D$$

$$\boxed{= -\frac{2}{5}t^{5/2} + Ct + D}$$

$$\boxed{1 = x(1) = -\frac{2}{5} + C + D}$$

given computed

$$\boxed{2 = x'(1) = -2 + C}$$

given computed

$$\boxed{C = 4}$$

$$1 = -\frac{2}{5} + 4 + D$$

$$1 = \frac{18}{5} + D$$

$$\boxed{-\frac{13}{5} = D}$$

$$\boxed{x(t) = -\frac{2}{5}t^{5/2} + 4t - \frac{13}{5}}$$

§1.3 Separable Eq'ts

$x' = x + t$

$$x' = f(x)g(t)$$

$$\Big\downarrow \text{div by } f(x)$$

$$\frac{1}{f(x)}\frac{dx}{dt} = g(t)$$

$$\Big\downarrow \int \dots dt$$

$$\int \frac{1}{f(x)}\frac{dx}{\cancel{dt}}\cancel{dt} = \int g(t)\,dt$$

$$\| $$

$$\int \frac{1}{f(x)}\,dx$$

$\Big\downarrow$ once computed, arrive at

$$\underbrace{\phi(t,x) = C}_{\text{"integral curves"}} \sim \text{"implicit soln"}$$

Ex: $\begin{cases} x' = \dfrac{t}{x} \\ x(2) = 1 \end{cases}$ $\quad f(x) = \frac{1}{x}$, $g(t) = t$

t-val, x-val

$$\int x\,dx = \int t\,dt$$

$$\frac{x^2}{2} = \frac{t^2}{2} + C \rightarrow x^2 = t^2 + \tilde{C} \qquad \tilde{C} = 2C$$

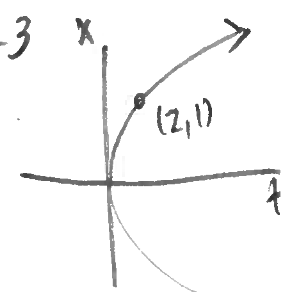
$$x^2 - t^2 = \tilde{C} \xrightarrow{\text{init cond}} 1^2 - 2^2 = \tilde{C}$$

$$1 - 4 = \tilde{C}$$

$$\tilde{C} = -3$$

$$x^2 = t^2 - 3$$

$$x = \pm\sqrt{t^2 - 3}$$



Ex 1.16:

$$x' = rx$$

$$\int \frac{1}{x}\,dx = \int r\,dt$$

$$\ln(x) = rt + C$$

$$x = \tilde{C}e^{rt}, \quad \tilde{C} = e^{C}$$

Sometimes: convenient to put bounds on $\int$ when doing sep'bl eqts

$$\int_{x_0}^{x} \frac{1}{f(x)}\,dx = \int_{t_0}^{t} g(t)\,dt$$

Ex 1.18:

$$\begin{cases} x' = \dfrac{2\sqrt{x}\,e^{-t}}{t} \\ x(1) = 4 \end{cases} \qquad x(t_0) = x_0$$

$$\frac{1}{\sqrt{x}} = x^{-1/2}$$

$$\int_{x_0}^{x} \frac{1}{\sqrt{s}}\,ds = \int_{t_0}^{t} \frac{2e^{-w}}{w}\,dw$$

$$\left.\frac{s^{1/2}}{1/2}\right|_{4}^{x} = 2\int_{1}^{t} \frac{e^{-w}}{w}\,dw$$

$$2\sqrt{x} - 4 = 2\int_{1}^{t} \frac{e^{-w}}{w}\,dw$$

$$x = \left(\int_{1}^{t} \frac{e^{-w}}{w}\,dw + 2\right)^{2}$$

$$x(1) = \left(\underbrace{\int_{1}^{1} \sim dt}_{=0} + 2\right)^{2} = 2^2 = 4 \checkmark$$