2021.01.13

$\int f = F + C$

$\int\int f = \int F + Ct$

$= \mathcal{F} + Ct + D$

A general class of 1st order ODE's

$$\boxed{x' = f(t, x)}$$

$x = x(t)$ unknown function

t time - indep. variable

Def: Has soln $x = x(t)$ on an interval $I = (a, b)$ if x is diff'bl on I + when substituted, satisfies the equation, i.e.

$$x'(t) = f(t, x(t)) \leftarrow \text{TRUE}$$

specific time t_0

Def: If we subject DE to a condition $x(t_0) = x_0$

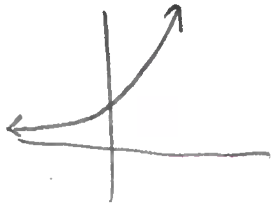
value of fnct.

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

"initial value problem" (IVP)

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg}{g^2}$$

Ex: Consider



A

$$\begin{cases} x' = 1 - x^2 \\ x(0) = 0 \end{cases}$$

Solved by

$$x(t) = \frac{e^{2t} - 1}{e^{2t} + 1}; \quad t \in \overset{I}{\downarrow}\mathbb{R}$$

Verify:

LHS

$$x'(t) = \frac{(e^{2t}+1)(2e^{2t}) - (e^{2t}-1)(2e^{2t})}{(e^{2t}+1)^2}$$

$$= \frac{\cancel{2e^{4t}} + 2e^{2t} - [\cancel{2e^{4t}} - 2e^{2t}]}{(e^{2t}+1)^2}$$

$$= \frac{4e^{2t}}{(e^{2t}+1)^2}$$

RHS

$$1 - x(t)^2 = 1 - \left(\frac{e^{2t} - 1}{e^{2t} + 1}\right)^2$$

$$= 1 - \frac{(e^{2t}-1)^2}{(e^{2t}+1)^2}$$

$$= \frac{(e^{2t}+1)^2 - (e^{2t}-1)^2}{(e^{2t}+1)^2}$$

$$= \frac{(\cancel{e^{4t}} + 2e^{2t} + \cancel{1}) - (\cancel{e^{4t}} - 2e^{2t} + \cancel{1})}{(e^{2t}+1)^2} = \frac{4e^{2t}}{(e^{2t}+1)^2}$$

match!
$\downarrow$
$x(t)$ solves $x' = 1 - x^2$

Init cond

$$x(0) = \frac{e^0 - 1}{e^0 + 1} = \frac{1-1}{1+1} = \frac{0}{2} = 0 \checkmark$$

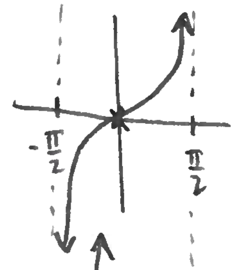
B

$\cos^2(t) + \sin^2(t) = 1$

$\downarrow$ div by $\cos^2(t)$

$1 + \tan^2(t) = \sec^2(t)$

$$\begin{cases} x' = 1 + x^2 \\ x(0) = 0 \end{cases}$$



Solved by

$$x(t) = \tan(t); \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Compute

$$\text{LHS} = x'(t) = \frac{d}{dt}[\tan(t)] = \frac{d}{dt}\left[\frac{\sin t}{\cos t}\right]$$

$$= \frac{\cos^2(t) + \sin^2(t)}{\cos^2(t)}$$

$$= \frac{1}{\cos^2(t)} = \sec^2(t)$$

$$\text{RHS} = 1 + x(t)^2 = 1 + \tan^2(t) = \sec^2(t)$$

match!
$\downarrow$
$x(t)$ solves $x' = 1 + x^2$

Exercises from p.10

(#3) Which of $x(t)=\frac{1}{t}$, $x(t)=\frac{2}{t}$, and $x(t)=\frac{1}{t-2}$ solve $x'=-x^2$?

Soln: (i) $x(t)=\frac{1}{t}$ → is a soln

$$\text{LHS} = x'(t) = -\frac{1}{t^2}$$

match!

$$\text{RHS} = -x(t)^2 = -\left(\frac{1}{t}\right)^2 = -\frac{1}{t^2}$$

(ii) $x(t)=\frac{2}{t}$ → not a soln

$$\text{LHS} = x'(t) = -\frac{2}{t^2}$$

do not match!

$$\text{RHS} = -x(t)^2 = -\left(\frac{2}{t}\right)^2 = -\frac{4}{t^2}$$

(iii) $x(t)=\frac{1}{t-2}$ → is a soln

$$\text{LHS} = x'(t) = -\frac{1}{(t-2)^2}$$

match!

$$\text{RHS} = -x(t)^2 = -\left(\frac{1}{t-2}\right)^2 = -\frac{1}{(t-2)^2}$$

(#4) Show $x(t) = e^{-t}\cos(t)$ solves

$$x'' + 2x' + 2x = 0$$

Soln: $x'(t) = -e^{-t}\cos(t) + e^{-t}(-\sin(t))$

$= -e^{-t}(\cos(t) + \sin(t))$

$x''(t) = e^{-t}(\cancel{\cos(t)} + \sin(t)) - e^{-t}(-\sin(t) + \cancel{\cos(t)})$

$= 2e^{-t}\sin(t)$

LHS $= x'' + 2x' + 2x$

$$= \underbrace{\cancel{2e^{-t}\sin(t)}}_{x''} - \underbrace{2e^{-t}(\cancel{\cos(t)} + \cancel{\sin(t)})}_{2x'} + \underbrace{2e^{-t}\cancel{\cos(t)}}_{2x}$$

$= 0 =$ RHS

(#8) Find m such that $x = t^m$ solves $t^2 x'' - 6x = 0$.

#ABD

↑

"guess"

$a \cdot b = 0$

$a = 0 \quad b = 0$

<u>Soln</u>: $x' = m t^{m-1}$

$x'' = m(m-1) t^{m-2}$

↓ plug in

$$t^2 \left[m(m-1) t^{m-2} \right] - 6t^m = 0$$

$$\left[m(m-1) - 6 \right] t^m = 0$$

↓

$$m(m-1) - 6 = 0$$

$$m^2 - m - 6 = 0$$

$$(m-3)(m+2) = 0$$

$m = 3 \quad m = -2 \quad \Rightarrow$ DE solved by

$$x(t) = t^3$$

$$x(t) = \frac{1}{t^2}$$

$\Rightarrow$ in fact, gen soln is

$$x(t) = c_1 t^3 + \frac{c_2}{t^2}$$

(#9?) Find m s.t. $x(t)=t^m$ solves $2tx'=x$

<u>Soln</u>: $x'(t)=mt^{m-1}$

Plug in:

$$2t(mt^{m-1})=t^m$$

$$2m\cancel{t^m}=\cancel{t^m}$$

$$2m=1$$

$$m=1/2 \longrightarrow$$ $x(t)=\sqrt{t}$ is a soln

$\Rightarrow$ general soln is

$$x(t)=c_1\sqrt{t}$$

§1.2.2 Growth-Decay models

e.g. radioactive decay, insect pops, chemicals released in nature, injection of medicine

$$\boxed{x' = -rx, \quad r > 0}$$ decay

$$\boxed{x' = rx, \quad r > 0}$$ growth