operators: $+, -, \times, \div$ operations: $\frac{d}{dt}, \int \sim dt$

$$\begin{array}{ccc} \text{Arithmetic} & \longrightarrow & \text{Calculus} \\ \downarrow & & \downarrow \\ \text{Algebra} & \longrightarrow & \text{Diff Eq} \end{array}$$

Algebra: relate unknowns using operations

$2x+1=3$

Diff Eq: relate unknown functions using calculus operations

Calculus Review

Derivatives of t^n, e^t, $\ln(t)$, $\sin(t)$, $\cos(t)$

$\frac{d}{dt}e^t = e^t$

$\log(t)$

Rules of derivatives: $f+g$, fg, $\frac{f}{g}$, $f \circ g$

chain rule

Fundamental thm of calculus: $\frac{d}{dt}\int_{t_0}^{t} f(s)\,ds = f(t)$

match!

$e^{a+b} = e^a e^b$

$e^{1-s} = e e^{-s}$

Key example

Consider $y(t) = \int_0^t e^{t-s} q(s)\,ds$

$$y(1) = \int_0^1 e^{1-s} q(s)\,ds = e\int_0^1 e^{-s} q(s)\,ds$$

$$y(5) = \int_0^5 e^{5-s} q(s)\,ds = e^5\int_0^5 e^{-s} q(s)\,ds$$

Claim: $y(t)$ solves "$\frac{dy}{dt} = y + q(t)$"

How to check?

Note:

$$y(t) = \underbrace{e^t}_{\text{funct of } t} \underbrace{\int_{t_0}^{t} e^{-s} q(s)\, ds}_{\text{funct of } t}$$

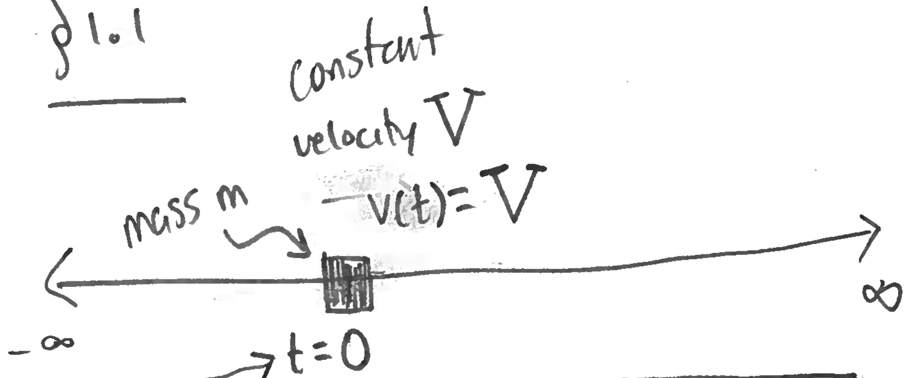
By product rule

$$\frac{dy}{dt} = \frac{d}{dt}[e^t] \int_{t_0}^{t} e^{-s} q(s)\,ds + e^t \frac{d}{dt} \int_{t_0}^{t} e^{-s} q(s)\, ds$$

$$= \underbrace{e^t \int_{t_0}^{t} e^{-s} q(s)\,ds}_{= y(t)} + e^t\left[e^{-t} q(t)\right]$$

$$= y(t) + q(t) \quad \checkmark\checkmark$$

§1.1



Now apply external force $\boxed{F = -kv(t) \quad (k>0)}$

drag coefficient

intuitively: object slows down (velocity decreases)

Newton's 2nd Law: $\boxed{F = \text{mass} \times \text{accel}}$

from calculus: $\boxed{a = v'}$

$$\boxed{mv' = -kv}$$ ← a diff eq

Equip w/ known measurement:

$$\begin{cases} mv' = -kv \\ v(0) = V \end{cases}$$

← a diff eq w/ extra info

← (initial condition)

Examples

A diff-eq is an equation that relates an unknown function to its derivative(s).

parameters: constants that are part of eqt

$$\theta'' + \sqrt{\frac{g}{\ell}}\sin(\theta) = 0 \qquad \text{(pendulum)}$$

$$Rq' + \frac{1}{C}q = \sin(\omega t) \qquad \text{(RC circuits)}$$

$$P' = rP\left(1 - \frac{P}{K}\right) \qquad \text{(logistic population model)}$$

$$T' = -h(T - Q) \qquad \text{(heating + cooling)}$$

$$t^2 y'' + t y' + (t^2 - \alpha^2) y = 0 \qquad \text{(Bessel's eqt)}$$

$$x' = \alpha \log\left(\frac{K}{x}\right) x \qquad \text{(Gompertz pop model)}$$

order 2

order 1

order: highest derivative that appears

linear: of form $y'' = p(t)y' + q(t)y + f(t)$