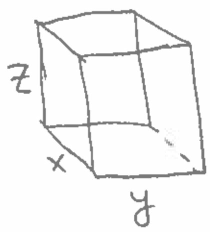
Ex: Given $12m^2$ of cardboard, maximize the volume of a box with no lid.

"constraint"

Soln: 

$$\text{Surface Area} = 2xz + 2yz + xy = 12$$

$$\text{Volume} = xyz \leftarrow \text{to maximize}$$

Solve constraint for a variable, say z:

$$z = \frac{12-xy}{2x+2y}$$

Plug constraint into $V = xyz$:

$$V = xy\left(\frac{12-xy}{2x+2y}\right) = \underbrace{\frac{1}{2}\left[\frac{12xy - x^2y^2}{x+y}\right]}_{f(x,y)}$$

Find critical points;

$$\begin{cases} f_x = \dfrac{1}{2}\left[\dfrac{(x+y)(12y-2xy^2)-(12xy-x^2y^2)(1)}{(x+y)^2}\right] \overset{\text{set}}{=} 0 & (i) \\ f_y = \dfrac{1}{2}\left[\dfrac{(x+y)(12x-2x^2y)-(12xy-x^2y^2)(1)}{(x+y)^2}\right] \overset{\text{set}}{=} 0 & (ii) \end{cases}$$

Since $x,y>0$ (else the volume makes no sense), $x+y>0$ so we can multiply both (i) and (ii) by $2(x+y)^2$ to get

$$\begin{cases} (x+y)(12y-2xy^2)-12xy+x^2y^2=0 & (iii) \\ (x+y)(12x-2x^2y)-(12xy)+x^2y^2=0 & (iv) \end{cases}$$

But

$$(x+y)(12y-2xy^2) = 12xy - 2x^2y^2 + 12y^2 - 2xy^3$$

and

$$(x+y)(12x-2x^2y) = 12x^2 - 2x^3y + 12xy - 2x^2y^2$$

Therefore (iii) and (iv) simplify to

$$\begin{cases} -x^2y^2+12y^2-2xy^3=0 & (v) \\ 12x^2-2x^3y-x^2y^2=0 & (vi) \end{cases}$$

Consider (v): factor out y^2 to get

$$y^2(-x^2+12-2xy)=0$$

$$(-x^2+12)^2 = x^4-24x^2+144$$

OR

$y^2=0 \Rightarrow y=0$ ~~useless!~~

$-x^2+12-2xy=0$

$$\boxed{y=\frac{-x^2+12}{2x}} \quad (\star)$$

plug into (vi)

$$12x^2-2x^3\left(\frac{-x^2+12}{2x}\right)-x^2\left(\frac{-x^2+12}{2x}\right)^2=0$$

mult by 4, simplify

$$\cancel{12x^2}+x^4-\cancel{12x^2}-\left(\frac{x^4-24x^2+144}{4}\right)=0$$

$$3x^4+24x^2-144=0$$

"quadratic in form" ⇝ let $w=x^2$

$$3w^2+24w-144=0$$

$$w=\frac{-24\pm\sqrt{24^2-4(3)(-144)}}{6}$$

$$=-4\pm\frac{48}{6}=-4\pm 8$$

~~$w=-4$~~ impossible since $w=x^2$

$\boxed{w=4}$

Since

$$x^2=w=4$$

$$\Downarrow$$

$x=\pm 2$, but x can't be negative

$\Rightarrow x=2$

$\Rightarrow$ by $(\star)$, $y=\dfrac{-2^2+12}{2(2)}=\dfrac{8}{4}=2 \Rightarrow (2,2)$ is crit pt

To finish problem, can compute (details omitted)

$$\frac{\partial^2 f}{\partial x^2} = -\frac{y^2(y^2+12)}{(x+y)^3} \qquad \frac{\partial^2 f}{\partial y^2} = -\frac{x^2(x^2+12)}{(x+y)^3}$$

and $$f_{xy} = f_{yx} = -\frac{xy(x^2+3xy+y^2-12)}{(x+y)^3}$$

so,

$$f_{xx}(2,2) = \frac{-4(16)}{4^3} < 0$$

$$f_{yy}(2,2) = \frac{-4(16)}{4^3}$$

$$f_{xy}(2,2) = f_{yx}(2,2) = \frac{-4(4+12+4-12)}{4^3}$$

Hence

$$\left.\begin{aligned} &D = f_{xx}f_{yy} - (f_{xy})^2 = \frac{3}{4} > 0 \\ \text{and} \quad &f_{xx} < 0 \end{aligned}\right\} \Rightarrow \underline{\text{local max}}$$