A theory is consistent if it does not prove a contradiction.

A theory is complete if every sentence (or its negation) has a proof in that theory.

(i.e. nothing is undecidable)

Earlier: we saw 1st order arithmetic (look at 8 April notes, p.2)

Add one more axiom:

Axiom (10) for any predicate P, the following is an axiom:

$$\big(P(0) \wedge \forall x\,(P(x) \rightarrow P(Sx))\big) \rightarrow \forall y\,(P(y))$$

"INDUCTION"

We call First Order Arithmetic + Axiom 10

"Peano arithmetic"

complete?

consistent?

Iteration - Fibonacci sequence

1, 1, 2, 3, 5, 8, ...

seed values

$$F_{n+1} = F_n + F_{n-1}$$
$$F_{n+2} = F_n + F_{n+1}$$

recursive defintion

self-referential

Factorial

$$4! = 4 \cdot 3 \cdot 2 \cdot 1$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

$$(n+1)! = (n+1)\underbrace{n(n-1)\ldots 2 \cdot 1}_{n!} = (n+1)\,n!$$

recursive

Ackermann function

$$Ack(x,y) = \begin{cases} y+1, & x=0 \\ Ack(x-1,1) & y=0 \\ Ack(x-1, Ack(x,y-1)) & \text{otherwise} \end{cases}$$

$x=0, y=0$

$$Ack(0,0) = 1$$

$x=0, y=1$

$$Ack(0,1) = 2$$

$\vdots$

$$Ack(0,y) = y+1$$

$$Ack(1,0) = Ack(1-1,0) = Ack(0,0) = 1$$

$x=1, y=0$

$$Ack(1,1) = Ack(0, Ack(0, \boxed{Ack(1,0)}))$$

$x=y=1$

$$= Ack(0,1) = 2$$

This function gets <u>very</u> large <u>very</u> fast:

$$\boxed{Ack(4,3) = 2^{2^{65,536}} - 3}$$

A <u>primitive recursive</u> function is a special kind of recursive function. Technical def too complicated for this class.

Fibonacci } primitive
factorial } recursive

Ackermann — not

<u>Theorem</u>: Any primitive recursive function can be expressed in Peano arithmetic.

<u>Gödel numbers</u>: a way to assign numbers to sentences in a theory

symbols	numbers
$0 \longleftrightarrow$	1
$\cdot \longleftrightarrow$	2
$+ \longleftrightarrow$	3
$= \longleftrightarrow$	4
$(\longleftrightarrow$	5
$) \longleftrightarrow$	6
$\forall \longleftrightarrow$	7
$\exists \longleftrightarrow$	8
$x \longleftrightarrow$	9

$y \longleftrightarrow 10$

$S0 \longleftrightarrow 11$

$SS0 \longleftrightarrow 12$

$\vdots$

$$S0 \cdot S0 = S0$$

$$\Big\downarrow \text{Gödel \#}$$

$$2^{11}\,3^{1}\,5^{2}\,7^{11}\,11^{1}\,13^{4}\,17^{11}\,19^{1} \leftarrow \underline{\underline{\text{BIG}}}$$

idea

encode abcd as

$$2^{\#(a)}\,3^{\#(b)}\,5^{\#(c)}\,7^{\#(d)}\,11^{\#(e)}\cdots$$

102
/ \
2 51
/ \
3 17

$51 = 3 \cdot 17$

$$\begin{array}{r} 17 \\ \times\ 3 \\ \hline 51 \end{array}$$

Super Gödel number

assigns a number to a proof

$2^{g_1}3^{g_2}\cdots$ where g_i is Gödel number of line i in proof

Proof of $S0+S0=SS0$ from 8 April p. 3:

$\forall y\,(S0+Sy=S(S0+y)) \longleftrightarrow 2^{7}\,3^{10}\,5^{5}\,7^{11}\,11^{1}\,13^{3}\,17^{11}\,19^{10}\,23\,29\,31 \ldots$ etc $= g_1$

$S0+S0=S(S0+0) \longleftrightarrow \# = g_2$

$S0+0=S0 \longleftrightarrow \# = g_3$

$S0+S0=SS0 \longleftrightarrow \# = g_4$

Super Gödel # of the proof is

$$2^{g_1}\,3^{g_2}\,5^{g_3}\,7^{g_4} \sim \text{incredibly large \#}$$

this was a proof of ϕ: $S0+S0=SS0$

$\ulcorner \phi \urcorner$

$Prf(m,n) \sim$ true if m is Super Gödel # of a proof of the sentence whose Gödel # is n

false otherwise

<u>Theorem</u> $P\mathring{r}f$ is primitive recursive

SSSSSS

If ϕ is a sentence, define

$$\mathrm{diag}(\phi) = (\exists y)\Big((y = \overset{\text{super Gödel \# of } \phi}{\ulcorner \phi \urcorner}) \wedge \phi\Big)$$

Define

$$Gdl(m,n) := Prf(m, \mathrm{diag}(m))$$

is true whenever m is Super Gödel # of a proof of the formula $\mathrm{diag}(m)$.

The <u>self-reference</u>: define

$$U_y = (\forall x)\, \neg Gdl(x,y)$$

$$\boxed{G = \mathrm{diag}(U_y)} \longleftarrow \text{Gödel sentence}$$

$$G = (\exists y)(y = \ulcorner U_y \urcorner \wedge U_y)$$

$$= \text{There exists } y \left(y = \ulcorner U_y \urcorner \wedge \underbrace{\forall x \; \neg Gdl(x, y)}_{} \right)$$

no number x exists such that x is the Super Gödel # of a proof of

$(\exists y)(y = \ulcorner U_y \urcorner \wedge U_y)$

Self-referential

Essentially: G says

"I am not provable."

if G had a proof

$\downarrow$

$\perp \implies$ G cannot have a proof

$\Downarrow$

G is true

uh oh! $\implies$ Peano arithmetic is not complete!

"Gödel's incompleteness theorem"