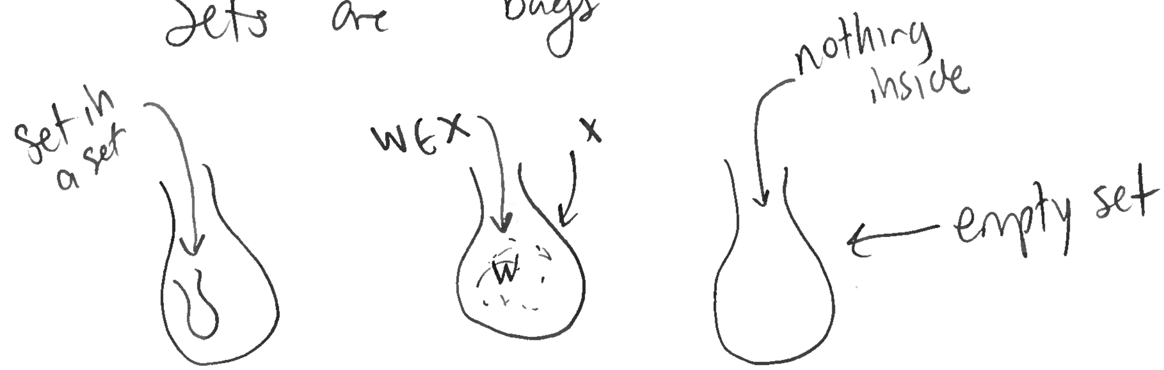
Sets are "bags"



<u>Extensionality</u> : says that sets which contain the same objects are the same set

<u>Comprehension</u> : given predicate P : can collect all objects which $\underbrace{\text{satisfy P}}_{\text{make P true}}$ into a set

An abbreviation:

define "$x \subseteq y$" to abbreviate

$$\forall z \, (z \in x \rightarrow z \in y)$$

in other words

$$x \subseteq y = \forall z \, (z \in x \rightarrow z \in y)$$

Prove Naive Set Theory $\vdash (x \subseteq y) \land (y \subseteq x) \rightarrow x = y$

1	$\forall z(z \in x \rightarrow z \in y) \land \forall z(z \in y \rightarrow z \in x)$	
2	$\forall z(z \in x \rightarrow z \in y)$	$\land E$ 1
3	$\forall z(z \in y \rightarrow z \in x)$	$\land E$ 1
4	$a \in x \rightarrow a \in y$	$\forall E$ 2
5	$a \in y \rightarrow a \in x$	$\forall E$ 3
6	$(a \in x \leftrightarrow a \in y) \leftrightarrow x = y$	$\forall E$ Ax 1
7	$a \in x$	
8	$a \in y$	$\rightarrow E$ 4,7
9	$a \in y$	
10	$a \in x$	$\rightarrow E$ 5,9
11	$a \in x \leftrightarrow a \in y$	$\leftrightarrow I$ 7–8, 9–10
12	$x = y$	$\leftrightarrow E$ 6, 11
13	$(\forall z(z \in x \rightarrow z \in y) \land \forall z(z \in y \rightarrow z \in x)) \rightarrow x = y$	$\rightarrow I$ 1–12

$(x \subseteq y \land y \subseteq x) \rightarrow x = y$

2

Let $P(x)$ be predicate $\neg(x=x)$.

Prove $\exists z \forall x (\neg(x \in z))$

1	$\exists z \forall x (x \in z \leftrightarrow \neg(x=x))$	Ax 2 with $P(x): \neg(x=x)$
2	$\forall x (x \in a \leftrightarrow \neg(x=x))$	
3	$b \in a \leftrightarrow \neg(b=b)$	$\forall$E 2
4	$b=b$	$=$I
5	$\neg(b=b)$	
6	$\bot$	$\neg$E 5, 4
7	$\neg\neg(b=b)$	$\neg$I 5–6
8	$b \in a$	
9	$\neg(b=b)$	$\leftrightarrow$E 3, 8
10	$b \in a \to \neg(b=b)$	$\to$I 8–9
11	$\neg(b \in a)$	MT 10, 7
12	$\forall x (\neg(x \in a))$	$\forall$I 11
13	$\exists z \forall x (\neg x \in z)$	$\exists$I 12
14	$\exists z \forall x (\neg x \in z)$	$\exists$E 1, 2–13

We give a special symbol for this important set : $\emptyset$

add this to our list of names

Barber paradox

Imagine a town that has exactly one barber.

The barber obeys property

"the barber shaves anyone who does not shave themselves"

Who in the town shaves the barber?

barber? → NO – because if barber shaved theirself, then the property says NO

not barber? → NO – again by property

Prove Naive Set Theory $\vdash \exists z(\emptyset \in z)$ 3

Let $P(x)$ be predicate $x = \emptyset$

1	$\exists z \forall x (x \in z \leftrightarrow x = \emptyset)$	Ax 2
2	$\quad \forall x (x \in a \leftrightarrow x = \emptyset)$	
3	$\quad \emptyset \in a \leftrightarrow \emptyset = \emptyset$	$\forall$E 2
4	$\quad \emptyset = \emptyset$	=I
5	$\quad \emptyset \in a$	$\leftrightarrow$E 3,4
6	$\quad \exists z (\emptyset \in z)$	$\exists$I 5
7	$\exists z (\emptyset \in z)$	$\exists$E 1, 2-6

Paradox

"This sentence is false"

if true $\Downarrow$ false

if false $\Downarrow$ true

Russel's paradox - shows naive set theory is contradictory!

Consider predicate $P(x)$: $\neg(x \in x)$

1	$\exists z \forall x (x \in z \leftrightarrow \neg(x \in x))$	Ax 2
2	$\quad \forall x (x \in a \leftrightarrow \neg(x \in x))$	
3	$\quad a \in a \leftrightarrow \neg(a \in a)$	$\forall$E 2
4	$\quad\quad a \in a$	
5	$\quad\quad \neg(a \in a)$	$\leftrightarrow$E 4
6	$\quad\quad \perp$	$\neg$E 5,4
7	$\quad \neg a \in a$	$\neg$I 4-6
8	$\quad a \in a$	$\leftrightarrow$E 3,7
9	$\quad \perp$	$\neg$E 7,8
10	$\perp$	$\exists$E 1, 2-9

Uh oh!

this contradiction is in the outermost level of proof.... so we derived $\perp$ from axioms of naive set theory!!

(Slides)

Banach Tarski